University of Helsinki / Department of Mathematics and Statistics
SCIENTIFIC COMPUTING
Exercise 04, 3.10.2011

`Problem sessions will be held on Monday at 16-18, B322.`

N.B. The files mentioned in the exercises (if any) are available on the course homepage

1. (a) Let X and Y be independent uniformly distributed random variables on $(0,1)$. As we know, samples of X can be generated by `x= rand(1,100);` for instance. Now it is a basic fact (this need not be proven) that the new random variables

$$U = \cos(2\pi X)\sqrt{-2\log Y}; \quad V = \sin(2\pi X)\sqrt{-2\log Y}$$

follow the normal distribution with parameters $(0,1)$, i.e. with mean 0 and variance 1. Use this so called Box-Müller method to generate 200 samples of normal distribution, plot the result with the command `hist`, compute the mean and standard deviation of the sample.

(b) The amplitude distribution of a signal sent by a mobile phone to a base station follows so called Rayleigh distribution. Suppose that X_1, X_2 are zero-mean normally distributed random variables with variance σ^2 and define a new random variable R by $R = \sqrt{X_1^2 + X_2^2}$. Then R follows the Rayleigh distribution. Generate 100 samples of a Rayleigh distribution and plot the histogram.

2. Suppose that $f\colon [a,b] \to [0,\infty)$ is continuous and that $0 \le f(x) \le M$ for all $x \in [a,b]$. Use the Monte Carlo method to approximate the value of

$$\int_a^b f(x)\,dx,$$

that is, choose m random points in $[a,b] \times [0,M]$ and compute the ratio p/m where p is the number of points below the graph of $f(x)$. Apply this method for the function

$$f(x) = \sum_{j=1}^{n} c_j(1 + \sin(d_j x))$$

FILE: ~/mme11/demo11/d04/d04.tex — 17. syyskuuta 2011 (klo 16.22).

in $[0,1]$ with $m = 10j$, $j = 10:10:100$ where $n = 4$, c=rand(1,n), d= 1+3*rand(1,n). Compare your result to the exact value

$$\int_a^b f(x)dx = (b-a)\sum_{j=1}^{n} c_j + \sum_{j=1}^{n}(c_j/d_j)(\cos(d_j * a) - \cos(d_j * b)),$$

see Problem 3/Exercise 2.

3. The ASCII codes of capital letters A,...,Z are 65,...,90. A simple ciphering method, so called Caesar cipher, is the following. Fix an integer $p \in [1, 25]$. Each letter is replaced by another, obtained by increasing its ASCII code by the constant p. (Note that we recycle: 91 corresponds to 65 i.e. after Z come A,B,C,...). The program hlp043.m shows how this happens. Use this idea to decipher the message:

```
Q C A D I H C S F U C G I A
```

4. We want to fit a model of the form $f(x) = ae^{bx}$ to the data set

x	1	3	4	6	9	15
y	4.0	3.5	2.9	2.5	2.75	2.0

where a and b are parameters to be determined from the data.

(a) For this purpose we introduce new transformed variables $X = x$, $Y = \log(y)$. Carry out this data transformation and print out the transformed variables.

(b) After the transformation the new model is $F(x) = \log f(x) = bx + \log a$. Apply the usual LSQ method to find b and $\log a$.

(c) Print the results in the following format

```
 x(i)      y(i)      Y(i)    a*exp(b*x(i))      y(i)-a*exp(b*x(i))
 1         4.0       ....
.......
15         2.0
```

and plot the data and the fitted curve in the same figure.

5. For a complex $n \times n$ matrix a let $P_i = \sum_{j=1,j\neq i}^{n} |a_{i,j}|$, $m_0 = \min\{|a_{i,i}| - P_i : i = 1, ..., n\}$, $m = \max\{m_0, 0\}$, $M = \max\{|a_{i,i}| + P_i : i = 1, ..., n\}$.

By Gerschgorin's theorem (recall Exercise 03) the eigenvalues λ_i of a satisfy

$$m \leq |\lambda_i| \leq M; \quad i = 1, ..., n$$

and it also follows that $m^n \leq D \leq M^n$, $D = |\det(a)|$. Set $m_1 = \min\{|\lambda_i| : i = 1, ..., n\}$ and $m_2 = \max\{|\lambda_i| : i = 1, ..., n\}$. Write a MATLAB script that experimentally confirms these statements, by printing out the test results in the following format

```
 n    m    m1    m2  M     D - m^n       M^n -D
```

Use random complex $n \times n$ matrices, n=5:5:50.

Repeat the experiment for the matrices a=2*n*eye(n)+rand(n,n)+i*rand(n,n).

6. The arithmetic-geometric mean $\mathrm{ag}(a, b)$ of two positive numbers $a > b > 0$ is defined as $\mathrm{ag}(a, b) = \lim a_n$, where $a_0 = a$, $b_0 = b$, and

$$a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0, 1, 2, \ldots$$

(a) Write a function, which takes two arguments (`double`), computes ag and returns the value (`double`).

(b) The hypergeometric function ${}_2F_1(a, b; c; x)$ is defined as a sum of the series,

$$\begin{aligned} {}_2F_1(a, b; c; x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \ldots \\ + \frac{a(a+1)\ldots(a+j-1)b(b+1)\ldots(b+j-1)}{c(c+1)\ldots(c+j-1)}\frac{x^j}{j!} + \ldots \quad . \end{aligned}$$

This hypergeometric series converges for $\operatorname{abs} x < 1$. Gauss proved in 1799 that there is a connection between the hypergeometric function and the arithmetic-geometric mean,

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; r^2) = \frac{1}{\mathrm{ag}(1, \sqrt{1 - r^2})}$$

for $0 < r < 1$. Tabulate the difference of the two sides of this identity for $r = 0.05k$, $k = 1, \ldots, 19$. Use the routine on the web-page to calculate the values of the ${}_2F_1$.