Linear Algebra and matrices II
Department of mathematics and statistics
Autumn 2011
Exercise sheet 6

These exercises are to be returned on Wed 14.12.2011 at 13.00, during the exam.

The central ideas in these exercises are

- Eigenvalues, eigenvectors and eigenspaces
- Diagonalaization

Exercise I

1. Find the eigenvalues for matrix

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. How can you see the eigenvalues straight from the matrix? Explain in your own words why it works.

3. Let us suppose that the 2×2-matrix A has eigenvalues 2 and 5. Let us also assume that $\bar{v}_2 = [1\ 0]^T$ and $\bar{v}_5 = [1\ 1]^T$ are eigenvectors corresponding to these eigenvalues. Find matrix A.

Exercise II

Let us suppose that the matrix $A \in \mathbb{R}^{n \times n}$ has an eigenvalue $\lambda \in \mathbb{R}$. The set

$$V_\lambda = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \lambda\bar{x}\}$$

is called the eigenspace corresponding to the eigenvalue λ of A. The eigenspace V_λ contains all the eigenvectors corresponding to λ and the zero vector as well.

4. Show that V_λ is a subspace of $\mathbb{R}^n$.

5. In exercise III of exercise sheet 5 we had the matrix

$$C = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Find the eigenspace corresponding to the eigenvalue 4 of C. (Your solution to exercise 5.III will be of help.)

6. Continuation from the previous exercise. Find the eigenspace corresponding to the eigenvalue 7 of C.
7. What is the dimension of the eigenspace of the previous exercise?

Exercise III

8. Show that the matrix
$$M = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$
is diagonalizable.
9. Since matrix M above is diagonalizable, there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}MP = D$. Find P and D.
10. Is matrix $B = \begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix}$ from exercise 5.III diagonalizable?
11. Is matrix C from exercise 5.III diagonalizable?
12. Back to the matrix M of exercise 8. Find the power M^{10}.

Exercise IV

Let
$$\mathcal{P}_2 = \{aX^2 + bX + c \mid a, b, c \in \mathbb{R}\}.$$
be the vector space of all the second degree polynomials with real coefficients. In this space sum and product are defined as follows: if $aX^2 + bX + c \in \mathcal{P}_2$, $a'X^2 + b'X + c' \in \mathcal{P}_2$ and $k \in \mathbb{R}$, then
$$(aX^2 + bX + c) + (a'X^2 + b'X + c') = (a + a')X^2 + (b + b')X + c + c'$$
and
$$k(aX^2 + bX + c) = kaX^2 + kbX + kc.$$
Let us define the linear mapping
$$L\colon \mathcal{P}_2 \to \mathbb{R}^2, \quad L(aX^2 + bX + c) = \begin{bmatrix} a - b \\ b + c \end{bmatrix}.$$

13. Which of the following polynomials are in the kernel of L
$$X + 1, \quad -X^2 + X, \quad X^2 + X - 1?$$
14. Which of the following polynomials are elements in $\mathrm{Im}(L)$:
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}?$$
15. Find the kernel $\mathrm{Ker}(L)$ and the image $\mathrm{Im}(L)$.

Exercise V

Let $A \in \mathbb{R}^{n \times n}$. Let us assume that $P \in \mathbb{R}^{n \times n}$ is an invertible matrix.

16. Show that every eigenvalue of A is also an iegenvalue for matrix $P^{-1}AP$.

    You can have a look to the hint at the end of this paper if you need it.

17. Show that every eigenvalue for $P^{-1}AP$ is also an eigenvalue for A.

18. What did you just show in the previous two exercises? Summarize the result into a theorem.

Exercise VI

Give your feedback! By giving feedback you'll add three exercises worth of extra points to your exercise points. Your feedback is important to us in order to improve our teaching. After the exam you can get via weboodi a link to your email address, through which you can get to fill in a feedback form.

Hint for exercise 16: Assume that λ is an eigenvalue for A and $\bar{v}$ the corresponding eigenvector. Then show that λ is also an eigenvalue for $P^{-1}AP$. Vector $P^{-1}\bar{v}$ helps.