Linear algebra and matrices II
Department of Mathematics and Statistics
Autumn 2011
Exercise sheet 5

Exercises due date: Mon 5.12.2011 at 17.00
Corrections due date: Fri 9.12.2011 at 17.00

The core ideas of these excercises are

- Injective and surjective linear mappings
- Isomorphisms
- Determinants
- Finding eigenvalues and eigenvectors

Exercise I

It can be proven that $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\det(A) \neq 0$.

Let

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{ja} \quad C = \begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}.$$

1. Basing on the determinant, decide whether A is invertible.
2. Basing on the determinant, decide whether B is invertible.
3. Basing on the determinant, decide whether C is invertible.
4. Is matrix $3CB^T$ invertible?

Exercise II

Let us consider the linear mapping $f \colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(\bar{x}) = [7x_1 \quad x_1 + x_2 \quad 3x_2 - x_1]^T$. In exercise 2 of exercise sheet 4 it was shown that f is an injection.

5. Show that f is not a surjection.
6. MWhy doesn't this contradict theorem 4.2.15?

Exercise III

Let $A \in \mathbb{R}^{n\times n}$. It can be proven that λ is an eigenvalue for A if and only if

$$\det(A - \lambda I) = 0.$$

By calculating $\det(A - \lambda I)$ we get a polynomial of the nth degree where λ is the variable. This polynomial is called the characteristic polynomial of A. The eigenvalues are therefore the zeroes of this polynomial. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix}$ ja $C = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

7. Find the eigevalues and the corresponding eigenvectors for A.

8. Find the eigevalues and the corresponding eigenvectors for B.

9. Find the eigevalues and the corresponding eigenvectors for C.

Exercise IV

10. Let $A \in \mathbb{R}^{n\times n}$. Show that A is invertible if and only if the number 0 is not an eigenvalue of A.

11. Let A be invertible and λ be and eigenvalue for A. Show that λ^{-1} is an eigenvalue for A^{-1}.

Exercise V

12. Define some isomorphism between the space $\mathbb{R}^2$ and the plane

$$T = \{\, \bar{x} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \,\}$$

(Hint: First find the generators for the plane.)

Exercise VI

Let us consider the linear mapping $L\colon \mathbb{R}^2 \to \mathbb{R}^2$, $L(\bar{x}) = [x_1{+}2x_2 \quad 4x_1{+}3x_2]^T$ and the vectors $\bar{a}_1 = [1\ 2]^T$, $\bar{a}_2 = [1\ \ -1]^T$ and $\bar{v} = [3\ 0]^T$.

13. Show that $S = \{\bar{a}_1, \bar{a}_2\}$ is a basis for vector space $\mathbb{R}^2$.

14. Find the coordinate vectors with respect to S of $\bar{a}_1$, $\bar{a}_2$ and $\bar{v}$ - in other words find $[\bar{a}_1]_S$, $[\bar{a}_2]_S$ and $[\bar{v}]_S$.

15. Find vectors $L(\bar{a}_1)$, $L(\bar{a}_2)$ and $L(\bar{v})$. What hare the coordinate vectors $[L(\bar{a}_1)]_S$, $[L(\bar{a}_2)]_S$ and $[L(\bar{v})]_S$?

16. Find a matrix B such that

$$B[\bar{a}_1]_S = [L(\bar{a}_1)]_S \quad \text{and} \quad B[\bar{a}_2]_S = [L(\bar{a}_2)]_S.$$

17. Calculate $B[\bar{v}]_S$. What do you notice?

Up to this point we have always written the matrix of a linear mapping with respect to the natural basis. Using another basis may prove itself useful too, though. It can be proven that for matrix B of the previous exercise

$$[L(\bar{v})]_S = B[\bar{v}]_S$$

for each vector $\bar{v}$ in the domain. Multiplying by B returns therefore the values for the mapping when all vectors are expressed as coordinate vectors with respect to basis S.

Notice that the columns of B are the image vectors of the vectors in basis S written with respect to S:

$$B = [[L(\bar{a}_1)]_S \quad [L(\bar{a}_2)]_S].$$

This result can be generalized to any linear mapping.

The matrix B is called the matrix of linear mapping L with respect to basis S. Such a matrix is indicated as $M(L; S \leftarrow S)$ in the lectures handouts.

Exercise VII

18–19. Do the MATLAB-exercises available on the course's web page - you can do them for instance in room C128. It is convenient to copy the commands straight from text of the exercises.