Linear algebra and matrices II
Department of mathematics and statistics
Autumn 2011
Exercise sheet 2

Exercises due date: Mon 14.11.2011 at 16.00
Corrections due date: Mon 18.11.2011 at 17.00

In these exercises we deal with

- Projections
- Finding an orthonormal base
- Orthogonal complement
- Cross product

Exercise I

1. Find the orthogonal projection of $\bar{v} = [4\ 4]^T$ on the space generated by vector $\bar{e}_1 = [1\ 0]^T$ and vector $\bar{a}_1 = [1\ \ -2]^T$.

2. Draw a picture of $\bar{v}$ and both projections.

Let V be an inner product space with subspace W. Let us suppose that W has the orthogonal basis $\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k\}$. Let $\bar{v} \in V$. Vector

$$\mathrm{proj}_W \bar{v} = \sum_{i=1}^{k} \frac{\langle \bar{v}, \bar{w}_i \rangle}{\langle \bar{w}_i, \bar{w}_i \rangle} \bar{w}_i$$

is the orthogonal projection of $\bar{v}$ on the space W.

We'll see later on that any finite-dimension subspace has always an orthogonal basis, whence the projection is always definable. Notice that using and orthogonal basis is indispensable, otherwise the vector $\bar{v} - \mathrm{proj}_W \bar{v}$ is not necessarily perpendicular to the plane W. We'll get back to this too later on.

If subspace W is generated by one vector $\bar{w} \neq \bar{0}$, we get the already known projection:

$$\mathrm{proj}_W \bar{v} = \mathrm{proj}_{\bar{w}} \bar{v} = \frac{\langle \bar{v}, \bar{w} \rangle}{\langle \bar{w}, \bar{w} \rangle} \bar{w}.$$

3. Let $\bar{w}_1 = [2\ 1\ 2]^T$ and $\bar{w}_2 = [2\ 0\ \ -2]^T \in \mathbb{R}^3$. Let $W = \mathrm{span}\{\bar{w}_1, \bar{w}_2\}$. Find the orthogonal projection of $\bar{a} = [-1\ 3\ 0]^T$ on subspace W.

4. Let V be an inner product space. Let us assume that $\bar{v}, \bar{w} \in V$ and $\bar{w} \neq \bar{0}$. Find vector's $\mathrm{proj}_{\bar{w}} \bar{v}$ norm, i.e. its length.

5. Calculate the norm for the previous exercise, with $\bar{w}$ as unit vector. In this case the absolute value of the dot product gives the length of the projection vector.

Exercise II

Let $\bar{a}_1 = [1\ 0\ 1]^T$, $\bar{a}_2 = [0\ 1\ 2]^T$, $\bar{a}_3 = [1\ -1\ 2]^T \in \mathbb{R}^3$.

6. Find an orthonormal basis for $\mathbb{R}^3$ using Gramin-Schmidtin method on the $\mathbb{R}^3$ basis $\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$.

7. Find the coordinates of vector $\bar{v} = [10\ 2\ 7]^T$ in the orthonormal basis for $\mathbb{R}^3$ you found.

8. Let $\bar{w}_1 = [2\ 1\ 2]^T$ and $\bar{w}_2 = [2\ 0\ -1]^T \in \mathbb{R}^3$. Let $W = \text{span}\{\bar{w}_1, \bar{w}_2\}$. Find the orthogonal projection of vector $\bar{a} = [2\ 3\ 4]^T$ on subspace W.

Exercise III

Sketch geometrically what the following subspaces $W \subset \mathbb{R}^3$ and their orthogonal complements $W^\perp$ look like

9. $W = \text{span}\{[1\ 1\ 1]^T\}$,

10. $W = \text{span}\{[2\ 0\ 0]^T, [0\ 0\ 3]^T\}$,

11. $W = \{\bar{0}\}$.

Tehtävä IV

Let V be an inner product space and $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$. We'll work on the subspace $W = \text{span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ generated by these vectors.

12. Let $\bar{v} \in V$ and $\langle \bar{v}, \bar{v}_i \rangle = 0$ for each $i \in \{1, 2, \ldots, k\}$. Show that $\langle \bar{v}, \bar{w} \rangle = 0$ for every $\bar{w} \in W$.

13. Explain in your own words the result you just got, without using any mathematical symbols.

Let $\bar{v} \in V$. Vector $\text{proj}_W \bar{v}$ belongs to the subspace W. Since we used an orthogonal basis in the definition of projection, it's straightforward proving that vector $\bar{v} - \text{proj}_W \bar{v}$ is orthogonal to every basis vector of W. From that it follows that $\bar{v} - \text{proj}_W \bar{v}$ is perpendicular to *every* vector of space W, that is $\bar{v} - \text{proj}_W \bar{v} \in W^\perp$.

14. Write vector $\bar{a}$ from exercise 3 as the sum of vectors from subspaces W and $W^\perp$.

Exercise V

Let $\bar{v} = [3\ 2\ 0]^T$ and $\bar{w} = [1\ 4\ 0]^T \in \mathbb{R}^3$.

15. Find the cross product of $\bar{v}$ and $\bar{w}$. (check chapter 3.5.)
16. Give an example of a vector which is orthogonal to the subspace generated by $\bar{v}$ and $\bar{w}$. Justify your answer.
17. Calculate the area of the parallelogram defined by $\bar{v}$ and $\bar{w}$.
18. Draw a triangle whose sides are $\bar{v}$, $\bar{w}$ and $\bar{w} + \bar{v}$. What is its area?
19. Draw a triangle whose sides are $\bar{v}$, $\bar{w}$ and $\bar{w} - \bar{v}$. What is its area?

Let

$$A = (2, -1, -2), \quad B = (1, -1, 1), \quad C = (-1, 3, -3) \quad \text{ja} \quad D = (-2, -1, 1).$$

be points in $\mathbb{R}^3$

20. Find the volume of the parallelepiped whose sides are $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$.