Linear algebra and matrices II
Department of mathematics and statistics
Autumn 2011
Exercise sheet 1

Exercises due date: Mon 7.11.2011 at 16.00
Corrections due date: Fri 11.11.2011 at 17.00

In these exercises we'll deal with

- Inner products, in particular the dot product
- Norm
- Angle between vectors and perpendicularity
- Orthogonal and orthonormal bases

Exercise I

Let $\bar{v} = [-1\ 2\ \ -2]^T$ and $\bar{w} = [5\ \ -2\ 1]^T \in \mathbb{R}^3$.

1. Calculate the dot product $\bar{v} \cdot \bar{w}$.
2. Calculate the norms $\|\bar{v}\|$ and $\|\bar{w}\|$ of vectors $\bar{v}$ and $\bar{w}$.
3. Find the angle between vectors $\bar{v}$ and $\bar{w}$.
4. Find the unit vector with the same direction as vector $\bar{v}$ (i.e. the vector whose norm is 1).

Exercise II

In the next exercises vectors belong to the space $\mathbb{R}^n$.

5. Let $\bar{a} = 3\bar{v} - \bar{w}$, $\bar{b} = \bar{v} + \bar{w}$, $\|\bar{v}\| = 2$, $\|\bar{w}\| = 3$ and $\bar{v} \cdot \bar{w} = -1$. Calculate $\bar{a} \cdot \bar{b}$.
6. Let us suppose that $\|\bar{v}\| = 2$, $\|\bar{w}\| = 3$ and $\bar{v} \cdot \bar{w} = 3$. Find $\|\bar{v} + 2\bar{w}\|$.

Exercise III

Let us assume that $\bar{v}, \bar{w} \in \mathbb{R}^n$ and $\bar{w} \neq \bar{0}$. Vector

$$\text{proj}_{\bar{w}}\bar{v} = \frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}\bar{w}$$

is the orthogonal projection of $\bar{v}$ onto the line with the same direction as $\bar{w}$. (We'll give a slightly more general definition of projection later on.)

7. Let $\bar{v} = [-3\ 1]^T$ ja $w = [1\ \ -1]^T$. Find the projection $\text{proj}_{\bar{w}}\bar{v}$.

8. Draw a picture of vectors $\bar{v}$, $\bar{w}$, $\text{proj}_{\bar{w}}\bar{v}$ and $\bar{v} - \text{proj}_{\bar{w}}\bar{v}$ from the previous exercise.

9. Let $\bar{v}, \bar{w} \in \mathbb{R}^n$. Show that vectors $\bar{w}$ and $\bar{v} - \text{proj}_{\bar{w}}\bar{v}$ are perpendicular (i.e. orthogonal) to each other. Compare this result to the picture drawn in exercise 8.

Exercise 4

Let $\bar{x} = [x_1\ x_2\ \ldots\ x_n]^T$, where $\bar{x} \in \mathbb{R}^n$.

Let us define $\bar{w} = [-1\ 3]^T$ and $\bar{v} = [9\ 2]^T$. Find out if vectors $\bar{v}$ and $\bar{w}$ orthogonal (that is perpendicular) if the dot product of space $\mathbb{R}^2$ is defined as follows:

10. $\langle \bar{w}, \bar{v} \rangle = w_1 v_1 + w_2 v_2$

11. $\langle \bar{w}, \bar{v} \rangle = 2w_1 v_1 + 3w_2 v_2$

12. $\langle \bar{w}, \bar{v} \rangle = \bar{w}^T A \bar{v}$, where
$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Exercise V

The basis of a vector space is said to be orthogonal if all of its vectors are perpendicular to each other. In addition, if the vectors' norm is one, the basis is said to be orthonormal. In the next exercises we'll see that it is easy to find vector coordinates in an orthonormal basis.

13. Let $T = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ be an orthonormal basis for vector space V. Let $\bar{w} = k_1\bar{a}_1 + k_2\bar{a}_2 + k_3\bar{a}_3$, where $k_1, k_2, k_3 \in \mathbb{R}$. Find $\langle \bar{w}, \bar{a}_i \rangle$ for each $\bar{a}_i \in T$.

14. The set $S = \{[1\ 1\ 1]^T,\ [-1\ 2\ -1]^T,\ [-2\ 0\ 2]^T\}$ is a basis for $\mathbb{R}^3$. Show that S is orthogonal.

15. Modify the basis S to make it orthonormal.

16. Find the coordinates of vector $[1\ 2\ 3]^T$ in the orthonormal basis you just got.

Exercise VI

Find out if the following are dot products in $\mathbb{R}^3$

17. $\langle \bar{x}, \bar{y} \rangle = 3x_1y_1 + x_2y_2 + 2x_3y_3$

18. $\langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2 - 2x_3y_3$.

for every $\bar{x}$, $\bar{y} \in \mathbb{R}^3$