Linear algebra and matrices I
Department of mathematics and statistics
Autumn 2011
Exercise sheet 6

Due date: Wed 19.10. at 13.00 during the course exam.

This week every exercise will grant you extra points, provided you genuinely attempted to solve them. All of the exercises of this week have to be returned at the same time during the course exam.

The central ideas of these exercises are

- Basis
- Dimension
- Coordinates

1. Let us suppose that the elements of the subspace W of $\mathbb{R}^3$ are, among the others, $[1\ 1\ 0]^T$ and $[4\ 0\ 0]^T$. Which of the following vectors are in subspace W as well? Back up your answer.

$$[2\ 0\ 0]^T, \quad [-30\ 1\ 0]^T, \quad [1\ 1\ 1]^T$$

2. Prove that the set $W = \{[c_1\ c_2\ 2c_2]^T \mid c_1, c_2 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

3. a) Let

$$\bar{a} = [2\ 0\ 1\ 4]^T, \quad \bar{b} = [1\ 2\ 0\ 0]^T, \quad \bar{c} = [3\ 1\ 0\ 2]^T, \quad \bar{d} = [4\ -4\ 3\ 12]^T.$$

Last week we showed that $\bar{d} \in \operatorname{span}\{\bar{a}, \bar{b}, \bar{c}\}$. Is vector $\bar{c}$ in the subspace $\operatorname{span}\{\bar{a}, \bar{b}, \bar{d}\}$?

   b) Let V be a vector space and $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. Show that if $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is free, then also $\{\bar{v}_1 - \bar{v}_3, \bar{v}_2 - \bar{v}_3, \bar{v}_3\}$ is.

4. Which of the following sets form a basis for $\mathbb{R}^4$? Which of them generate $\mathbb{R}^4$? (Hint: 2.5.6 & 2.5.14.)

   a) $\{[-1\ 2\ 4\ 0]^T,\ [-3\ 1\ 1\ 0]^T,\ [0\ 0\ 0\ 2]^T\}$
   b) $\{[1\ 0\ 0\ 1]^T,\ [0\ 1\ 0\ 0]^T,\ [1\ 1\ 1\ 1]^T,\ [1\ 1\ 1\ 0]^T\}$
   c) $\{[1\ 1\ 0\ 0]^T,\ [2\ 2\ 0\ 0]^T,\ [0\ 0\ 1\ 0]^T,\ [0\ 0\ 0\ 1]^T\}$
   d) $\{[1\ 1\ 0\ 0]^T,\ [1\ 1\ 1\ 1]^T,\ [0\ 0\ 1\ 0]^T,\ [0\ 0\ 0\ 1]^T,\ [2\ 2\ 0\ 0]^T\}$.

Let V be a vector space and $S = \{v_1, v_2, \ldots, v_n\}$ a basis for it. Then a vector $\bar{v} \in V$ can be written as a linear combination of the elements of the basis:

$$\bar{v} = a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n \quad \text{for some } a_1, a_2, \ldots, a_n \in \mathbb{R}.$$

Coefficients $a_1, a_2, \ldots, a_n$ define vector $\bar{v}$ completely. If the coefficients are known, then we know exactly what vector $\bar{v}$ is like.

The coefficients $a_1, a_2, \ldots, a_n$ are called the *coordinates* of v and the vector $[\bar{v}]_S = [a_1\ a_2\ \cdots\ a_n]^T$ is called the *coordinate vector* for $\bar{v}$ with respect to the basis S. Notice that the coordinates of a vector depend always on the basis chosen.

5. Let $\bar{v}_1 = [1\ 1]^T$, $\bar{v}_2 = [-2\ 3]^T$, $\bar{e}_1 = [1\ 0]^T$ and $\bar{e}_2 = [0\ 1]^T$.

   a) Show that the set $S = \{\bar{v}_1, \bar{v}_2\}$ is a basis for $\mathbb{R}^2$.

   b) Find the coordinate vector for vector $\bar{a} = [1\ 6]^T$ with respect to S.

   c) Find the vector $\bar{b} \in \mathbb{R}^2$ whose coordinate vector with respect to basis S is $[3\ \ -1]^T$. In other words find $\bar{b} \in \mathbb{R}^2$ for which $[\bar{b}]_S = [3\ \ -1]^T$.

   d) Draw two pictures of vectors $\bar{a}$ and $\bar{b}$: one where the coordinate axes are parallel to the base vectors $\bar{e}_1$ and $\bar{e}_2$ of the normal basis, and another one where the coordinate axes are parallel to the vectors in S.

6. Let us assume that $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6$ are six vectors from $\mathbb{R}^5$. Choose the right answer in each of the following cases. Remember to justify your answer.

   a) Vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6$ *generate* $\mathbb{R}^5$ / *could generate* $\mathbb{R}^5$ / *don't generate* $\mathbb{R}^5$.

   b) Vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6$ *are* / *could be* / *are not* linearly independent.

   c) Five randomly chosen vectors among $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6$ *always form a basis* / *could form a basis* / *never form a basis* for $\mathbb{R}^5$.

7. Find a basis for the subspace

   $$W = \{\, [x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 + 2x_2 = 0 \text{ and } x_2 - x_3 = 0 \,\}.$$

   of $\mathbb{R}^3$. What is the dimension of subspace W?

8. Give feedback for the course! If you give feedback you will be granted five exercises worth of bonus points. Your feedback is important to us, as it helps us improving our teaching. Include your student number when giving feedback so that you can be granted the bonus points. Feedback questions concern the exam as well, so it will be possible to answer the questionnaire only after the exam; you can access the link to the questionnaire through weboodi.