Linear algebra and matrices I
Department of mathematics and statistics
Autumn 2011
Exercise sheet 5

Due date: Mon 10.10. at 18.00.
Corrections due date: Fri 14.10. at 17.00.

Just like we did last week, only starred exercises will be checked, and they have to be correct for you to get points out of them. You can correct starred exercises and return them again if needed. You get points out of all other exercises as long as you honestly attempted to solve them.

Core ideas of these exercises are

- Generation
- Independence
- Basis

In exercises 1–5 we work on vectors of the vector space $\mathbb{R}^2$

$$\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \bar{v}_4 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \bar{v}_5 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and on the matrices

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

1. Draw vectors $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ as points in the reference frame and connect them by a line in numbered order.

Matrices can be thought of as mappings which map a vector onto another. To do so we use matrix multiplication. If for instance $A \in \mathbb{R}^{2\times 2}$, we can form the mapping

$$f_A \colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad f_A(\bar{v}) = A\bar{v}.$$

So vector $\bar{v}$ is mapped onto vector $A\bar{v}$.

2. * Calculate vectors $f_B(\bar{v}_1)$, $f_B(\bar{v}_2)$, $f_B(\bar{v}_3)$, $f_B(\bar{v}_4)$ and $f_B(\bar{v}_5)$, where B is as defined above.

3. Draw a picture of the vectors $f_B(\bar{v}_1), f_B(\bar{v}_2), \ldots, f_B(\bar{v}_5)$ and connect the points to each other in numbered order. What do you notice?

4. Find the vectors $f_C(\bar{v}_1), f_C(\bar{v}_2), \ldots, f_C(\bar{v}_5)$, where C is as defined above.

5. Draw a picture of the vectors $f_C(\bar{v}_1), f_C(\bar{v}_2), \ldots, f_C(\bar{v}_5)$ and connect the points to each other in numbered order. What do you notice?

6. Say in your own words why does an elementary matrix always have an inverse matrix.

In exercises 7–9 we deal with the line ℓ, which passes through $A = (1, 2)$ and $B = (-4, 5)$.

7. * Draw the corresponding position vector of vector $\overline{AB}$. Use the picture to decide if the vector (that is the point corresponding to it) on the line ℓ.

8. Choose two points on ℓ and draw their corresponding position vectors.

9. * Sum together the vectors you chose in the last exercise and draw their corresponding position vector. Use the picture to decide if their sum is on the line ℓ.

In the last two exercises we noticed that if a line doesn't pass through the origin, then it doesn't have all of the properties of a subspace. In fact if a line (or a plane) doesn't cross the origin it doesn't have any subspace property.

10. * Find the subspace of $\mathbb{R}^2$ generated by vector $[1\ 2]^T$. Draw a picture of it.

In exercises 11–13 we deal with $\mathbb{R}^4$ vectors

$$\bar{a} = [2\ 0\ 1\ 4]^T, \quad \bar{b} = [1\ 2\ 0\ 0]^T, \quad \bar{c} = [3\ 1\ 0\ 2]^T, \quad \bar{d} = [4,\ -4\ 3\ 12]^T.$$

We want to find out whether the vector $\bar{d}$ is a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$. In other words we have to verify if equation

$$\bar{d} = x_1\bar{a} + x_2\bar{b} + x_3\bar{c}$$

is fulfilled for some $x_1, x_2, x_3 \in \mathbb{R}$.

11. Turn the aforementioned vector equation into a system of equations.

12. Solve the system of equations you got in the last exercise.

13. * Is vector $\bar{d}$ a linear combination of vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$? In other words: does vector $\bar{d}$ belong to the subspace generated by vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$? That is does $\bar{d} \in \operatorname{span}\{\bar{a}, \bar{b}, \bar{c}\}$ hold?

14. * Let us suppose that $[b_1\ b_2]^T \in \mathbb{R}^2$. Show that $[b_1\ b_2]^T$ can be expressed as a linear combination of vectors $[1\ 5]^T$ and $[-5\ \ -1]^T$, i.e. that vectors $[1\ 5]^T$ and $[-5\ \ -1]^T$ generate the vector space $\mathbb{R}^2$.

15. * After you had struggled the whole morning with the proof of a linear algebra theorem, your cat jumped on your tea cup and split it all over your notes (which you can find at the end of this exercise sheet). Unfortunately, the text is only partially readable. Fill in the blanks and return the page along with your other answers.

16. Let $\bar{v}_1 = [1\ 5]^T$ and $\bar{v}_2 = [-5\ \ -1]^T$. Let us suppose that $x_1, x_2 \in \mathbb{R}$ are such that $x_1\bar{v}_1 + x_2\bar{v}_2 = 0$. Prove that $x_1 = 0$ and $x_2 = 0$.

17. * Is the set $\{[1\ 5]^T,\ [-5\ -1]^T\}$ in $\mathbb{R}^2$ free (that is linearly independent)? Draw a picture of the situation.

18. * Is the set $\{[3\ 1]^T,\ [-6\ -2]^T\}$ in $\mathbb{R}^2$ free (that is linearly independent)? Draw a picture of the situation.

Is the set $\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ linearly independent in the next exercises? Give a careful explanation.

19. * $\bar{a}_1 = [1\ 0\ 2]^T$, $\bar{a}_2 = [3\ 1\ 0]^T$ ja $\bar{a}_3 = [1\ 1\ 1]^T$

20. * $\bar{a}_1 = [4\ 2\ 1]^T$, $\bar{a}_2 = [4\ \ -3\ 2]^T$ ja $\bar{a}_3 = [0\ 5\ \ -1]^T$

21. * Let V be a vector space and $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \in V$. Show that if $v_4 \in \operatorname{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, then the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is not free.

22. * Prove that the vectors $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ in exercise 19 form a basis for $\mathbb{R}^3$.

23. Write vectors $[1\ -2\ 4]^T \in \mathbb{R}^3$ using the basis from last exercise. Can you find more than one way to do it?

24. Let U and W be subspaces of vector space V. Show that the intersection

$$U \cap W = \{\bar{v} \in V \mid \bar{v} \in U \text{ and } \bar{v} \in W\}$$

is still a subspace of V.

25. Subspaces of the space $\mathbb{R}^3$ are, among others, lines and planes passing through the origin. Think about how two such subspaces' intersection could be like.

Claim: Let V be a vector space and $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in V$. Then $\operatorname{span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ is a subspace of V.

Proof: Let $W = \operatorname{span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$.

Since the vector space $\mathbb{R}^3$ contains all the linear combinations of its vectors, we know that $W \subset \mathbb{R}^3$.

i) Let $\bar{v}, \bar{w} \in W$. Then

$$\bar{v} = a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n$$

and

$$\underline{\hspace{6cm}}$$

for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$. We notice that

$$\begin{aligned} \bar{v} + \bar{w} &= (a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n) + (b_1\bar{v}_1 + b_2\bar{v}_2 + \cdots + b_n\bar{v}_n) \\ &= \underline{\hspace{8cm}} \\ &= (a_1 + b_1)\bar{v}_1 + (a_2 + b_2)\bar{v}_2 + \cdots + (a_n + b_n)\bar{v}_n, \end{aligned}$$

therefore $\underline{\hspace{2cm}}$.

ii) Let us suppose that $\bar{v} \in W$ and $\underline{\hspace{2cm}}$. Now $\bar{v} = a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n$ for some $\underline{\hspace{5cm}}$. Then

$$\begin{aligned} \underline{\hspace{1.5cm}} &= r(a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_n\bar{v}_n) \\ &= \underline{\hspace{6cm}}, \end{aligned}$$

therefore $r\bar{v} \in W$.

iii) We see that $\bar{0} = \underline{\hspace{6cm}}$, from which it follows that $\bar{0} \in W$.

So W is a subspace of V.