Linear algebra and matrices I
Department of mathematics and statistics
Autumn 2011
Exercise sheet 4

Due date: Mon 3.10 at 18.00.
Corrections due date: Fri 7.10. at 17.00.

NOTE! This week only the exercises marked with a star sign will be checked. Your solution to such exercises has to be correct for you to get points out of them. If they are not, you are allowed to make corrections and return them again. A honest attempt to solve non-starred exercises will be enough to get points from them.

Central ideas of these exercises are

- Vectors
- Vector spaces
- Subspaces
- Generation

In the next exercises we'll deal with the set $\mathbb{R}^{n\times 1}$ of $n \times 1$ matrices. It's customary to mark $\mathbb{R}^{n\times 1} = \mathbb{R}^n$ and refer to the elements of the set as column vectors or just vectors. Column vectors can be geometrically illustrated in many different ways in a coordinate system.

i) Column vectors can be thought of as points in the coordinate system. For instance the vector $[a\ b]^T$ in $\mathbb{R}^2$ can be identified with the point (a, b).

ii) A column vector can be drawn as an arrow whose starting point is the origin and the end point is the point the vector is identified with in the coordinate system. As an example, the vector $[a\ b]^T$ in $\mathbb{R}^2$ can be represented as an arrow with starting point in the origin and end point in (a, b). This vector is called the position vector of the point (a, b).

iii) We can think of a vector as an arrow with a length and a direction in the coordinate system. The position of the arrow is not important. For example the vector $[a\ b]^T$ in $\mathbb{R}^2$ can be identified with any other arrow having the same direction and length as the position vector of $[a\ b]^T$.

1. Consider the following vectors as points in a xy-coordinate system and draw them:
$$\bar{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \bar{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \bar{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

2. Calculate using the vectors from the previous exercise:
$$\bar{v} + \bar{u}, \ \bar{v} + \bar{w}, \ 3\bar{u}, \ (-1)\bar{v}, \ \bar{v} + (-1)\bar{v}.$$

3. Draw the position vectors corresponding to $\bar{v}$, $\bar{u}$, $\bar{v} + \bar{u}$ and $3\bar{u}$ where $\bar{v}$ and $\bar{u}$ are as above.

4. Think of the following $\mathbb{R}^1$ vectors as points of the real line and draw their corresponding position vectors:
$$[2], \ [-4].$$

5. Let $\overline{PQ}$ be a vector whose starting point is $P = (-1, 2)$ and the end point is $Q = (0, 4)$. Draw a picture of the vector. Draw then another picture of the vector, this time make it start from the origin. What $\mathbb{R}^2$ element does the vector correspond to?

Vector addition can be illustrated by moving vectors around in the coordinate system. Visually, addition is made by attaching the first vector's end point to the second one's starting point. The starting point of the resulting vector is then the first vector's starting point, while its end point is the second vector's end point.

6. Use the method described above to represent the vectors you calculated in exercise 2, $\bar{v} + \bar{u}$ and $\bar{v} + (-1)\bar{v}$.

Let ℓ be the line passing through $A = (2, -1, 4)$ and $B = (1, -1, 3)$.

7. * Find vectors $\bar{a} = \overline{OA}$ and $\bar{v} = \overline{AB}$.

8. * Find the parametric equation of ℓ (see page 44.)

9. Give an example of some point of ℓ which is neither A nor B.

10. Does ℓ pass through the origin?

Let T be the plane passing through $A = (-2, 3, 9)$, $B = (-1, 4, 10)$ and $C = (-5, 5, 17)$.

11. Find vectors $\bar{a} = \overline{OA}$, $\bar{v} = \overline{AB}$ and $\bar{w} = \overline{AC}$

12. * Find the parametric equation of T.

13. Does T pass through the origin?

14. * Give a counterexample to prove that the following statement does not hold.

$$\text{Let } A \in \mathbb{R}^{m\times n} \text{ and } B, C \in \mathbb{R}^{n\times l}. \text{ If } AB = AC, \text{ then } B = C.$$

Find out if statements in exercises 15–17 are facts or fluffs. Justify your answer.

15. * Fact or fluff? Multiplying a matrix's row by $1/\sqrt{2}$ is not an elementary row operation.

16. * Fact or fluff? If in a system of linear equations there are as many unknowns as equations, the system as exactly one solution.

17. Fact or fluff? If in a system of linear equations there are more unknowns than equations, the system doesn't have a unique solution.

The set $\mathbb{R}^n$ of column vectors is an example of vector space (see def. 2.2.1). A vector space with real coefficients is, in a nutshell, a set where items can be summed together and multiplied by a real number. Several rules hold for these operations. In addition in a vector space there exists a zero vector and every vector has an opposite vector.

Vector spaces have subspaces (see def. 2.3.1). For example the lines and planes passing through the origin are subspaces.

18. * Show that the set $U = \left\{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 = x_2 \text{ ja } x_3 = 0\right\}$ is a subspace of $\mathbb{R}^3$.

Why are subspaces defined the way they are defined? A subspaces is a given vector space's subset, but this is not enough. A subspace has also to be a vector space. So in a subspace it must be possible to sum items together and to multiply them by a scalar. Furthermore a subspace must contain the zero vector. The definition of subspaces comes from these conditions, which are enough to guarantee that what we're dealing with is a subspace. From the definition it follows indeed that a subspaces contains always the opposite vector of every of its vectors: we can get it by multiplying the vector by -1. In addition, calculation rules of a vector space are automatically in force in its subspaces, because all the vectors of the subspace are also items of the original vector space.

In the following cases, check if the set U is a subspace of $\mathbb{R}^3$. (Hint: Try to find a couple of examples of items of U.)

19. * $U = \{ [x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_1 = x_2^2 \}$

20. * $U = \{ [x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_3 = x_1 + x_2 \}$

21. * $U = \{ [x_1\ x_2\ x_3]^T \in \mathbb{R}^3 \mid x_3 = x_1 \text{ or } x_3 = x_2 \}$.

22. Is the line ℓ in exercise 8 a subspace of the vector space $\mathbb{R}^3$?

23. Is the plane T in exercise 12 a subspace of the vector space $\mathbb{R}^3$?

24. Continuation from previous exercises. If the line ℓ was a subspace, give an example of some vectors generating it (see pages 39–40). Do the same thing for the plane T.

Let $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

25. The matrix A is a vector in the vector space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices. What is the zero vector of this vector space?

26. * What are vectors $\frac{1}{2}A$ and $-A$?

27. * What kind of subspace does vector A generate? In other words, what is $\text{span}(A)$?

28. Give an example of a subspace of the vector space $\mathbb{R}^{2\times 2}$ to which B belongs and C doesn't.

29. * If both B and C belong to a subspace of the vector space $\mathbb{R}^{2\times 2}$, must the identity matrix I belong to that subspace, too?

30. Solve the MATLAB-exercise at:
`https://wiki.helsinki.fi/display/mathstatKurssienLisasivut/Linis+MATLAB`
Just the answer will be enough. The link is available also on the course's website.