Linear Algebra and Matrix Theory I
Department of Mathematics and Statistics
Fall 2011
Exercises 3

Deadline for the exercises: Monday 26.9 18:00.
Deadline for the corrected exercises: Friday 30.9. 17:00.

The main contents of these exercises are:

- Defining an inverse matrix
- Elementary matrices
- Homogeneous systems of linear equations

You don't have to write the assignments to your answer sheet.

1. Reduce
$$\begin{bmatrix} 2 & 0 & 1 & | & 1 & 0 & 0 \\ 3 & 1 & 0 & | & 0 & 1 & 0 \\ 2 & -1 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$
into unit upper triangular form. (You don't have to worry about the lines in the matrix. They are there to create a separation between two parts of the matrix, which will turn out be a useful method in the future.)

2. Continue the previous exercise and reduce the matrix to a so called reduced row echelon form. (see definition 1.5.1 from the lecture material and the Gauss-Jordan elimination method).

3. Suppose $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$. Is the matrix A invertible? If yes, define its inverse matrix A^{-1}. (Make use of the previous exercise and the calculation method 1.6.7 in the lecture material.)

4. Suppose $B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Is the matrix B invertible? If yes, define its inverse matrix A^{-1}.

In exercises 5–8 we consider the matrix:

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ -8 & 0 & 1 \end{bmatrix}.$$

5. Find the inverse matrix of the matrix C.

6. Create a chart of all the elementary row operations in the order you used them in the previous exercise, include also the elementary matrices corresponding them.

7. Assume that the elementary matrices (in the order you used them) are $E_1, \ldots, E_k$. Calculate the product $E_k \cdots E_1 C$.

8. How can you define the inverse matrix C^{-1} by using the elementary matrices of the previous exercise?

In exercises 9–13 we consider the system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 &= -1 \\ 4x_1 - x_2 + x_3 &= 5. \end{cases}$$

9. Verify that

$$\begin{cases} x_1 &= 5 \\ x_2 &= 8 \\ x_3 &= -7 \end{cases}$$

is a solution of the system.

10. Write the homogeneous system corresponding the system of linear equations (see p. 22 of the lecture material).

11. Solve the homogeneous system of the previous exercise by using Gaussian or Gauss-Jordan elimination method.

12. Form the solution of the original system by using the solutions found in exercises 9 and 11 (see exercise set 2: exercises 26 and 27, or lecture material Theorem 1.5.13.).

13. With Gaussian elimination method we find that the solution of the original system is:

$$\bar{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{4}{7} \\ -\frac{9}{7} \\ 1 \end{bmatrix}, \quad \text{where } s \in \mathbb{R}.$$

Compare this result with the solution of the previous exercise. Are they the same?

14. Your friend proved the following result of linear algebra:

> Suppose that $A \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times 1}$ and $B \in \mathbb{R}^{n\times 1}$. If the matrix A is invertible, then the equation $AX = B$ has exactly one solution.

The proof looked like the following:

$$\begin{aligned} &AX = B \\ &A^{-1}AX = A^{-1}B \\ &X = A^{-1}B \\ \\ &A(A^{-1}B) = B \quad \text{OK!} \end{aligned}$$

The proof is rather hard to read. Rewrite it with a better style. Remember to write down all the needed assumptions. In a good answer a list of each step is not enough, you need to write words, implications ($\Rightarrow$) or other proper notations between them.

In exercises 15 and 16 we consider the set of matrices:

$$S = \left\{ \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \ : \ c_1, c_2 \in \mathbb{R} \right\}.$$

15. Show that the sum of any two elements of S belongs to S. In other words, if $A, B \in S$ then $A + B \in S$. Remember to write your proof with care and list all your assumptions etc.

16. A continuation to the previous exercise. Show that for all $A \in S$ and $r \in \mathbb{R}$ it is true that $rA \in S$. Remember to write your proof with care and list all your assumptions etc.

In exercises 17–19 we consider the system:

$$\begin{cases} x_1 - \ \ x_2 + 2x_3 &= 1 \\ 2x_1 - \ \ x_2 + ax_3 &= 2 \\ -x_1 + 2x_2 + \ \ x_3 &= b \end{cases}.$$

17. Write the augmented matrix corresponding the system and reduce it to a unit upper triangular form.

18. Define those real numbers a for which there exists exactly one solution.

19. In what cases does the system not have a unique solution? Define those $b \in \mathbb{R}$ for which there then does exist a solution. What is the solution?