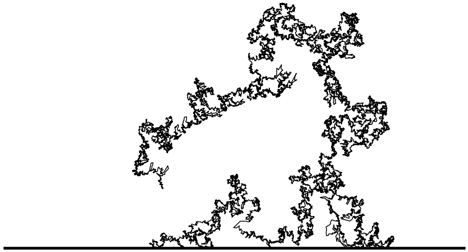
Introduction to Mathematical Physics: Schramm–Loewner evolution

Antti Kemppainen

University of Helsinki

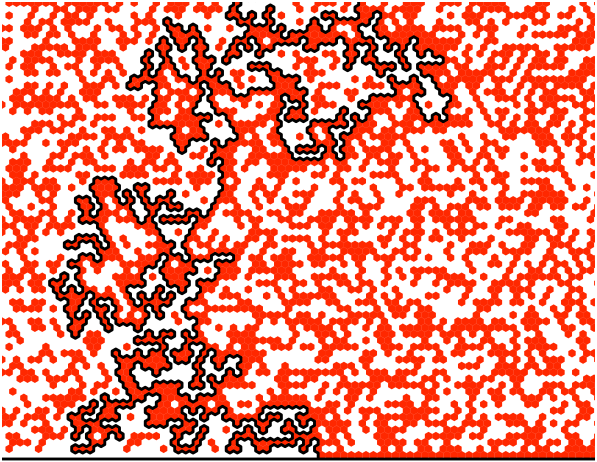
September 8th 2011



Introduction

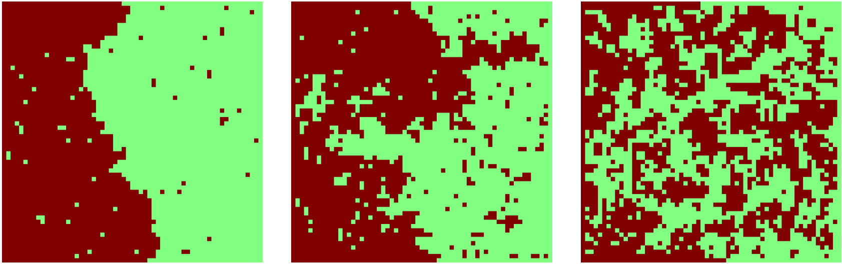
Example of a model of statistical physics: Percolation

- The simplest model to formulate. I.i.d. coin flips.
- Each site is either *open* or *closed*. A parameter $p \in [0, 1]$ is the probability that a given site is open. The state of each site is chosen independently.



Example of a model of statistical physics: Ising model

- The simplest model with "physical" interaction. Ferromagnetism.
- Each site has a state which is ± 1. A parameter $\beta > 0$ is the inverse temperature. The state of the system is $\sigma \in \{-1, +1\}^{\text{sites}}$
- Energy: $H(\sigma) = -\sum_{i,j \text{ neighbours}} \sigma_i \cdot \sigma_j$
- Propability distribution: $\mathbb{P}(\sigma) = \frac{1}{Z} \exp(-\beta H(\sigma))$



Origin of conformal invariance in physics

- In ordered phase there are typical length scales:
  $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...

Origin of conformal invariance in physics

- In ordered phase there are typical length scales:
  $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...
- As $\beta \searrow \beta_c$, $\xi_1 \to \infty$
- "$\xi_1 = \infty$" $\approx$ no typical length scale $\approx$ scale invariance

Origin of conformal invariance in physics

- In ordered phase there are typical length scales:
  $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...
- As $\beta \searrow \beta_c$, $\xi_1 \to \infty$
- "$\xi_1 = \infty$" $\approx$ no typical length scale $\approx$ scale invariance
- Under some additional hypothesis, the continuum limit should be invariant under transformations which are locally dilations.
- If a transformation is locally combination of rotation, translation and scaling, it is *conformal* $\Leftrightarrow$ preserves angles.

Origin of conformal invariance in physics

- In ordered phase there are typical length scales:
  $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...
- As $\beta \searrow \beta_c$, $\xi_1 \to \infty$
- "$\xi_1 = \infty$" $\approx$ no typical length scale $\approx$ scale invariance
- Under some additional hypothesis, the continuum limit should be invariant under transformations which are locally dilations.
- If a transformation is locally combination of rotation, translation and scaling, it is *conformal* $\Leftrightarrow$ preserves angles.
- Conformal symmetry is very strong in two dimensions.

Origin of conformal invariance in physics

- In ordered phase there are typical length scales: $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...
- As $\beta \searrow \beta_c$, $\xi_1 \to \infty$
- "$\xi_1 = \infty$" $\approx$ no typical length scale $\approx$ scale invariance
- Under some additional hypothesis, the continuum limit should be invariant under transformations which are locally dilations.
- If a transformation is locally combination of rotation, translation and scaling, it is *conformal* $\Leftrightarrow$ preserves angles.
- Conformal symmetry is very strong in two dimensions.
    - In physics: $\Rightarrow$ Conformal field theories

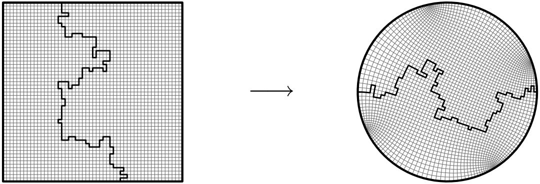
Origin of conformal invariance in physics

- In ordered phase there are typical length scales:
  $\xi_1 =$ correlation length, $\xi_2 =$ size of fluctuations of interface, ...
- As $\beta \searrow \beta_c$, $\xi_1 \to \infty$
- "$\xi_1 = \infty$" $\approx$ no typical length scale $\approx$ scale invariance
- Under some additional hypothesis, the continuum limit should be invariant under transformations which are locally dilations.
- If a transformation is locally combination of rotation, translation and scaling, it is *conformal* $\Leftrightarrow$ preserves angles.
- Conformal symmetry is very strong in two dimensions.
  - In physics: $\Rightarrow$ Conformal field theories
  - In mathematics: $\Rightarrow$ Schramm–Loewner evolution

Schramm's principle

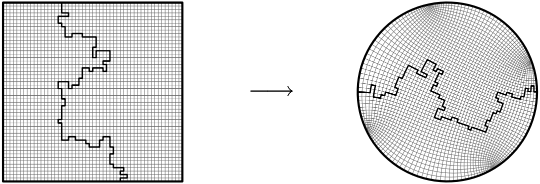
- Let $\mathbb{P}^{U,a,b}$ be the law of a random curve in the domain $U \subset \mathbb{C}$ connecting the points a and b.
- Schramm's principle: if $(\mathbb{P}^{U,a,b})$ satisfies

- Let $\mathbb{P}^{U,a,b}$ be the law of a random curve in the domain $U \subset \mathbb{C}$ connecting the points a and b.
- Schramm's principle: if $(\mathbb{P}^{U,a,b})$ satisfies *conformal invariance*

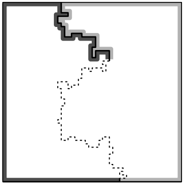


Schramm's principle

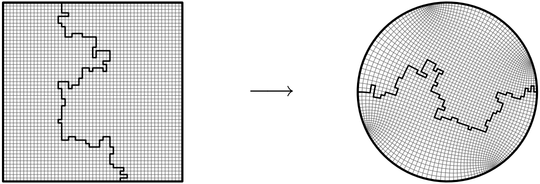
- Let $\mathbb{P}^{U,a,b}$ be the law of a random curve in the domain $U \subset \mathbb{C}$ connecting the points a and b.
- Schramm's principle: if $(\mathbb{P}^{U,a,b})$ satisfies *conformal invariance*



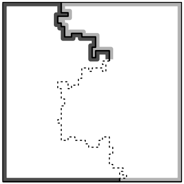
and *domain Markov property*



- Let $\mathbb{P}^{U,a,b}$ be the law of a random curve in the domain $U \subset \mathbb{C}$ connecting the points a and b.
- Schramm's principle: if $(\mathbb{P}^{U,a,b})$ satisfies *conformal invariance*

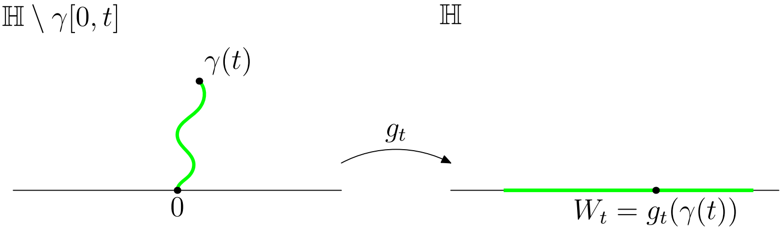


and *domain Markov property*

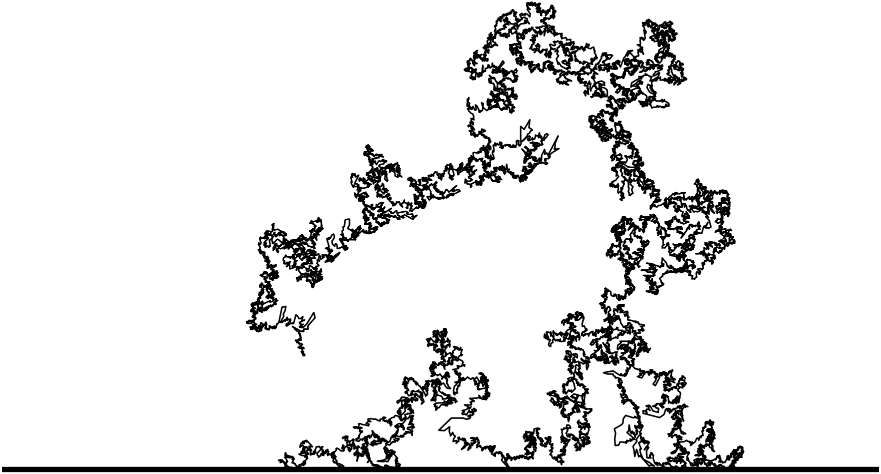


then $(\mathbb{P}^{U,a,b})$ is Schramm–Loewner evolution SLE(κ) for some $\kappa \geqslant 0$.

- Random motion on the real axis, *stochastic process*
- Family of conformal maps associated to the process, *Loewner chain*



$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z) - W_t}$$



Contents

The lectures will cover (tentatively) the following list topics:

1. Stochastic analysis
2. Conformal mappings and complex analysis
3. Loewner equation
4. Schramm's principle
5. Definition of Schramm–Loewner evolution
6. Example calculations using SLE
7. SLE is a random curve
8. Fractal properties of SLE
9. Symmetries of SLE ("time reversal" symmetry)
10. On the connection between statistical physics and SLE

Measure theory and probability = Probability theory

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$
- Monotone convergence theorem: If f_n are measurable functions such that $0 \leqslant f_n \nearrow f$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$
- Dominated convergence theorem: If f_n are measurable functions and $f = \lim_{n\to\infty} f_n$ exists almost everywhere and $\exists g \geqslant 0$ such that $|f_n| \leqslant g$ for all n and $\int g \mathrm{d}\mu < \infty$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$.

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$
- Monotone convergence theorem: If f_n are measurable functions such that $0 \leqslant f_n \nearrow f$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$
- Dominated convergence theorem: If f_n are measurable functions and $f = \lim_{n\to\infty} f_n$ exists almost everywhere and $\exists g \geqslant 0$ such that $|f_n| \leqslant g$ for all n and $\int g \mathrm{d}\mu < \infty$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$.
- $L^p(\mu)$ space: Measurable f is in $L^p(\mu)$ if $\int |f|^p \mathrm{d}\mu < \infty$. Notation: $\|f\|_p = (\int |f|^p \mathrm{d}\mu)^{1/p}$.

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$
- Monotone convergence theorem: If f_n are measurable functions such that $0 \leqslant f_n \nearrow f$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$
- Dominated convergence theorem: If f_n are measurable functions and $f = \lim_{n\to\infty} f_n$ exists almost everywhere and $\exists g \geqslant 0$ such that $|f_n| \leqslant g$ for all n and $\int g \mathrm{d}\mu < \infty$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$.
- $L^p(\mu)$ space: Measurable f is in $L^p(\mu)$ if $\int |f|^p \mathrm{d}\mu < \infty$. Notation: $\|f\|_p = (\int |f|^p \mathrm{d}\mu)^{1/p}$.
- Product measures: If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then their product space is $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ where $X \times Y$ is Cartesian product, $\mathcal{A} \times \mathcal{B}$ the σ-algebra generated by $A \times B$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $\mu \times \nu$ the unique extension of $A \times B \mapsto \mu(A)\nu(B)$.

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$
- Monotone convergence theorem: If f_n are measurable functions such that $0 \leqslant f_n \nearrow f$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$
- Dominated convergence theorem: If f_n are measurable functions and $f = \lim_{n\to\infty} f_n$ exists almost everywhere and $\exists g \geqslant 0$ such that $|f_n| \leqslant g$ for all n and $\int g \mathrm{d}\mu < \infty$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$.
- $L^p(\mu)$ space: Measurable f is in $L^p(\mu)$ if $\int |f|^p \mathrm{d}\mu < \infty$. Notation: $\|f\|_p = (\int |f|^p \mathrm{d}\mu)^{1/p}$.
- Product measures: If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then their product space is $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ where $X \times Y$ is Cartesian product, $\mathcal{A} \times \mathcal{B}$ the σ-algebra generated by $A \times B$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $\mu \times \nu$ the unique extension of $A \times B \mapsto \mu(A)\nu(B)$.
- Fubini's theorem: Let $f \in \mathcal{A} \times \mathcal{B}$. If $f \geqslant 0$ or $\int |f| \mathrm{d}(\mu \times \nu) < \infty$ then $\int_X (\int_Y f \mathrm{d}\nu) \mathrm{d}\mu = \int_{X\times Y} f \mathrm{d}(\mu \times \nu) = \int_Y (\int_X f \mathrm{d}\mu) \mathrm{d}\nu$.

Measure theory

- $(X, \mathcal{A})$ a measurable space: X is set, $\mathcal{A}$ its σ-algebra
- measurable function f, measure μ, integral $\int f \mathrm{d}\mu$
- Monotone convergence theorem: If f_n are measurable functions such that $0 \leqslant f_n \nearrow f$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$
- Dominated convergence theorem: If f_n are measurable functions and $f = \lim_{n\to\infty} f_n$ exists almost everywhere and $\exists g \geqslant 0$ such that $|f_n| \leqslant g$ for all n and $\int g \mathrm{d}\mu < \infty$, then $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$.
- $L^p(\mu)$ space: Measurable f is in $L^p(\mu)$ if $\int |f|^p \mathrm{d}\mu < \infty$. Notation: $\|f\|_p = (\int |f|^p \mathrm{d}\mu)^{1/p}$.
- Product measures: If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then their product space is $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ where $X \times Y$ is Cartesian product, $\mathcal{A} \times \mathcal{B}$ the σ-algebra generated by $A \times B$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $\mu \times \nu$ the unique extension of $A \times B \mapsto \mu(A)\nu(B)$.
- Fubini's theorem: Let $f \in \mathcal{A} \times \mathcal{B}$. If $f \geqslant 0$ or $\int |f| \mathrm{d}(\mu \times \nu) < \infty$ then $\int_X (\int_Y f \mathrm{d}\nu) \mathrm{d}\mu = \int_{X \times Y} f \mathrm{d}(\mu \times \nu) = \int_Y (\int_X f \mathrm{d}\mu) \mathrm{d}\nu$.
- Radon–Nikodym theorem: if ν is a signed measure and μ is a measure on $(X, \mathcal{A})$ and $\nu(A) = 0$ whenever $\mu(A) = 0$, then exist $g \in \mathcal{F}$ such that $\nu(A) = \int_A g \mathrm{d}\mu$.

Probability theory

- A probability space is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}$ is a probability measure, i.e., $\mathbb{P}(\Omega) = 1$. Ω "outcomes", $\mathcal{F}$ "events"
- A random variable is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$. H-valued random variable is a measurable function $X : \Omega \to H$ (H is a measurable space).
- The expected value of X is $\mathbb{E}(X) = \int X \mathrm{d}\mathbb{P} \in [-\infty, \infty]$, which makes sense when $X \geqslant 0$ or when either $\int X^+ \mathrm{d}\mathbb{P} < \infty$ or $\int X^- \mathrm{d}\mathbb{P} < \infty$, $X = X^+ - X^-$.
- $L^p(\mathbb{P})$ space: $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p} < \infty$. By Hölder inequality, $\|X\|_p \leqslant \|X\|_q$ for $1 \leqslant p \leqslant q$ and hence $L^q(\mathbb{P}) \subset L^p(\mathbb{P})$.

Probability theory

- A probability space is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}$ is a probability measure, i.e., $\mathbb{P}(\Omega) = 1$. Ω "outcomes", $\mathcal{F}$ "events"
- A random variable is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$. H-valued random variable is a measurable function $X : \Omega \to H$ (H is a measurable space).
- The expected value of X is $\mathbb{E}(X) = \int X \mathrm{d}\mathbb{P} \in [-\infty, \infty]$, which makes sense when $X \geqslant 0$ or when either $\int X^+ \mathrm{d}\mathbb{P} < \infty$ or $\int X^- \mathrm{d}\mathbb{P} < \infty$, $X = X^+ - X^-$.
- $L^p(\mathbb{P})$ space: $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p} < \infty$. By Hölder inequality, $\|X\|_p \leqslant \|X\|_q$ for $1 \leqslant p \leqslant q$ and hence $L^q(\mathbb{P}) \subset L^p(\mathbb{P})$.
- Independence: sub-σ-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{F}$ are *independent* if

$$\mathbb{P}(A_1, A_2, \ldots, A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \ldots \cdot \mathbb{P}(A_n) \quad \text{for } A_k \in \mathcal{A}_k.$$

Random variables $X_1, X_2, \ldots, X_n$ are independent if σ-algebras $\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)$ are independent

$$\Leftrightarrow \mathbb{P}(\{X_1 \in B_1\} \cap \{X_2 \in B_2\} \cap \ldots \cap \{X_n \in B_n\}) = \mathbb{P}(X_1 \in B_1) \cdot \mathbb{P}(X_2 \in B_2) \cdot \ldots \cdot \mathbb{P}(X_n \in B_n) \quad \text{for } B_k \in \mathcal{B}_{\mathbb{R}}.$$

Probability theory

- A probability space is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}$ is a probability measure, i.e., $\mathbb{P}(\Omega) = 1$. Ω "outcomes", $\mathcal{F}$ "events"
- A random variable is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$. H-valued random variable is a measurable function $X : \Omega \to H$ (H is a measurable space).
- The expected value of X is $\mathbb{E}(X) = \int X \mathrm{d}\mathbb{P} \in [-\infty, \infty]$, which makes sense when $X \geqslant 0$ or when either $\int X^+ \mathrm{d}\mathbb{P} < \infty$ or $\int X^- \mathrm{d}\mathbb{P} < \infty$, $X = X^+ - X^-$.
- $L^p(\mathbb{P})$ space: $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p} < \infty$. By Hölder inequality, $\|X\|_p \leqslant \|X\|_q$ for $1 \leqslant p \leqslant q$ and hence $L^q(\mathbb{P}) \subset L^p(\mathbb{P})$.
- Independence: sub-σ-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{F}$ are *independent* if

$$\mathbb{P}(A_1, A_2, \ldots, A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \ldots \cdot \mathbb{P}(A_n) \quad \text{for } A_k \in \mathcal{A}_k.$$

  Random variables $X_1, X_2, \ldots, X_n$ are independent if σ-algebras $\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)$ are independent

$$\begin{aligned} &\Leftrightarrow \mathbb{P}(\{X_1 \in B_1\} \cap \{X_2 \in B_2\} \cap \ldots \cap \{X_n \in B_n\}) \\ &\quad = \mathbb{P}(X_1 \in B_1) \cdot \mathbb{P}(X_2 \in B_2) \cdot \ldots \cdot \mathbb{P}(X_n \in B_n) \quad \text{for } B_k \in \mathcal{B}_{\mathbb{R}}. \end{aligned}$$

- Notation: $\mathbb{E}(X; E) = \int_E X \mathrm{d}\mathbb{P} = \int \mathbb{1}_E X \mathrm{d}\mathbb{P}$.