1. Suppose that g_t is the solution of the Loewner equation with a continuous driving term $(W_t)_{t\in\mathbb{R}_+}$. Show that the inverse maps $f_t = g_t^{-1}$ satisfy the equation

$$\partial_t f_t(z) = -f_t'(z)\frac{2}{z-W_t}, \qquad f_0(z) = z. \tag{1}$$

2. Let g_t and W_t be as above and let $a_k(t)$ be the coefficients in the expansion

$$g_t(z) = z + \sum_{k=1}^{\infty} a_k(t) z^{-k}.$$

In the Loewner equation, expand $2/(g_t(z) - W_t)$ in powers of z near ∞ and find a (closed) system of differential equations satisfied by $a_k(t)$ for $k = 1, 2, \ldots, 5$. Integrate those equations to give explicit formulas for $a_k(t)$ in terms of the driving term.

3. Fix $0 < \alpha < 1$.

(a) Show that $z \mapsto (1-z)^\alpha$ has the expansion $(1-z)^\alpha = 1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \ldots$ near 0.

(b) Let $a < b$ be real numbers. Define a continuous function $\phi_{a,b} : \overline{\mathbb{H}} \to \mathbb{C}$ which is holomorphic in $\mathbb{H}$ by setting

$$\phi_{a,b}(z) = (z-a)^{1-\alpha}(z-b)^\alpha$$

where the branches are chosen so that $\phi_{a,b}(x) > 0$ for $x > b$. Show that $\phi_{a,b}$ maps $\mathbb{R}$ onto $\mathbb{R} \cup \{rw_0 : 0 < r \leq 1\}$ where

$$w_0 = e^{i\pi\alpha}(1-\alpha)^{1-\alpha}\alpha^\alpha(b-a).$$

(c) Find for each $t \in \mathbb{R}_+$, $a_t < b_t$ such that $f_t = \phi_{a_t,b_t}$ has an expansion of the form

$$f_t(z) = z - \frac{2t}{z} + \ldots$$

near ∞. Verify that f_t satisfies the differential equation (1) with $W_t = K_\alpha\sqrt{t}$, where K_α is some constant. Deduce that the driving term of curve $\gamma : t \mapsto c_\alpha e^{i\alpha\pi}\sqrt{t}$ is W_t where $c_\alpha > 0$ is a constant such that γ is parametrized with the half-plane capacity. Show also that $\phi_{a,b} : \mathbb{H} \to \mathbb{C}$ is a conformal map for any $a < b$.

4. To motivate Problems 4 and 5, we first state the Carátheodory kernel theorem. Let $w_0 \in \mathbb{C}$ be given. Let U_n be a sequence of domains with $w_0 \in U_n \subset \mathbb{C}$. Then we say that the sequence U_n converges to U as $n \to \infty$ with respect to w_0 in the sense of *kernel convergence* if

- either $U = \{w_0\}$ or U is a domain not equal to $\mathbb{C}$ with $w_0 \in U$ such that some neighborhood for any $w \in U$ is in U_n for large n,
- for any $w \in \partial U$ there exist $w_n \in \partial U_n$ such that $w_n \to w$ as $n \to \infty$.

Let $f_n : \mathbb{D} \to U_n$ be the conformal onto map with $f_n(0) = w_0$ and $f_n'(0) > 0$. If $U = \{w_0\}$, then set $f(z) = w_0$, otherwise let $f : \mathbb{D} \to U$ be the conformal onto map with $f(0) = w_0$ and $f'(0) > 0$. The *Carátheodory kernel theorem* states that $U_n \to U$ as $n \to \infty$ with respect to w_0 is equivalent to $f_n \to f$ uniformly in compact subsets of $\mathbb{D}$.

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Let $T > 0$. Show that for each $\varepsilon > 0$, there exists $\delta_0, \delta_1 > 0$ such that the following holds. If $z_0 \in \mathbb{H}$, $W \in C([0,T])$ and z_t is the solution of the equation

$$\dot{z}_t(z) = \frac{2}{z_t - W_t}, \ t \in [0,T]$$

and they satisfy the condition

$$\inf_{t\in[0,T]} |z_t - W_t| \geq \varepsilon,$$

then for any $\tilde{z}_0 \in B(z_0, \delta_0)$ and $\tilde{W} \in C([0,T])$ with $\|W - \tilde{W}\|_\infty < \delta_1$ the solution of

$$\dot{\tilde{z}}_t(z) = \frac{2}{\tilde{z}_t - \tilde{W}_t}$$

exists for all $t \in [0,T]$ and it satisfies the condition

$$\inf_{t\in[0,T]} |\tilde{z}_t - \tilde{W}_t| \geq \varepsilon/2.$$

Furthermore, the inequality

$$|z_t - \tilde{z}_t| \leq e^{At}\left(|z_0 - \tilde{z}_0| + A^{-1}\left(1 - e^{-At}\right)\|W^{(n)} - W\|_{\infty,[0,T]}\right)$$

holds where $A = 4\varepsilon^{-2}$.

Hint. You might want to use the following version of *Gronwall's lemma*: if $f(t)$ is nonnegative, differentiable function on $[0,T]$ which satisfies

$$f'(t) \leq Af(t) + B$$

where A and B are nonnegative constants, then

$$f(t) \leq e^{At}\left(f(0) + \frac{B}{A}\left(1 - e^{-At}\right)\right).$$

5. (a) Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls that have a continuous driving term $(W_t)_{t\in[0,T]}$. Show that there exists a sequence of simple curves $\gamma^{(n)} : [0,T] \to \mathbb{C}$ in the upper half-plane parametrized with the half-plane capacity such that the conformal maps $g_t^{(n)}$ associated to $\gamma^{(n)}$ converge to g_t uniformly in both $t \in [0,T]$ and $z \in \{z \in \mathbb{H} : \operatorname{dist}(z, K_T) \geq \varepsilon\}$ for every $\varepsilon > 0$.

Hint. Use the other exercises of this problem sheet.

(b) Let K be a hull. Show that there is a sequence of hulls K_n such that $\partial(\mathbb{H} \setminus K_n)$ is a curve and that g_{K_n} converges uniformly to g_K in the set $\{z \in \mathbb{H} : \operatorname{dist}(z, K) \geq \varepsilon\}$ for each $\varepsilon > 0$.