UNIVERSITY OF HELSINKI

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Schramm–Loewner evolution, Fall 2011
Problem Sheet 8 (Nov 8)

1. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. For $x \in \mathbb{R}$ define $\tau_x = \inf\{t \in \mathbb{R}_+ : B_t = x\}$.

(a) Let $a < 0 < b$. Show that $\mathbb{P}(\tau_a \wedge \tau_b < \infty) = 1$ by considering $\mathbb{P}(B_t < a \text{ or } B_t > b)$.

(b) Apply the optional stopping theorem to $\mathbb{E}(B_{\tau_a\wedge\tau_b})$ and find the probabilities of the events $\{B_{\tau_a\wedge\tau_b} = a\}$ and $\{B_{\tau_a\wedge\tau_b} = b\}$ for $a < 0 < b$.

(c) Show that for all $x \in \mathbb{R}$, $\tau_x < \infty$ almost surely.

2. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion and let $(X_t)_{t\in\mathbb{R}_+}$ be a Brownian motion with drift defined by $X_t = \sigma B_t + \mu t$ where $\sigma > 0$ and $\mu \geq 0$ are constants. Let $x > 0$ and let $\tau_x = \inf\{t \in \mathbb{R}_+ : X_t = x\}$.

(a) Let $\lambda > 0$. Consider the process $(M_t)_{t\in\mathbb{R}_t}$ defined by

$$M_t = \mathbb{E}\left(\exp(-\lambda\tau_x) | \mathcal{F}_t\right).$$

Show that it is a martingale. Let $f(x) = \mathbb{E}\left(\exp(-\lambda\tau_x)\right)$. Show that $M_t = f(x-X_t)\exp(-\lambda t)$ for $t < \tau_x$.

(b) Assuming that f is smooth, find a second order differential operator L such that $Lf = 0$. *Hint.* A semimartingale $Y_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}B_s$ is a local martingale if and only if almost surely $u_t = 0$ for all t.

(c) Solve the equation $Lg = 0$ with the following boundary conditions: $g(0) = 1$ and $g(x)$ stays bounded as $x \to \infty$. Use the optional stopping theorem to show that $\tilde{M}_t = g(x - X_t)\exp(-\lambda t)$ indeed satisfies $\tilde{M}_t = M_t$, $t < \tau_x$. This justifies the smoothness assumption.

(d) Is $\mathbb{E}(\tau_x) < \infty$? Find the inverse Laplace transform of $\lambda \mapsto \mathbb{E}\left(\exp(-\lambda\tau_x)\right)$ (if you can).

3. Let $\mathbb{P}^z$ be the law of a complex Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ sent from z. Let K be a hull and let $\tau_K = \inf\{t \in \mathbb{R}_+ : B_t \in \mathbb{R} \cup K\}$. Show that for any $z \in \mathbb{H} \setminus K$, $\mathbb{P}^z(\tau_K < \infty) = 1$ and

$$a_1(K) = \lim_{y\nearrow\infty} y\, \mathbb{E}^{iy}\left(\operatorname{Im} B_{\tau_K}\right).$$

4. Let $\delta \in \mathbb{R}$, $x > 0$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. A *Bessel process with dimension δ started from x* is the solution $(X_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{\delta - 1}{2X_t}\mathrm{d}t, \qquad X_0 = x.$$

The theorem about SDEs from the lecture notes can be applied to show that the solution exists and is unique at least up to the stopping time

$$\tau = \sup\left\{t \in \mathbb{R}_+ : \inf_{s\in[0,t]} X_s > 0\right\}.$$

Denote the law of $(X_t)_{t\in[0,\tau)}$ by $\mathbb{P}^x$.

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(a) Let $\lambda > 0$. Show that the process $Y_t = \lambda X_{t/\lambda^2}$ is a Bessel process of dimension δ started from λx.

(b) For any $y > 0$, let $\sigma_y = \inf\{t \in [0, \tau) : X_t = y\}$ (here the infimum of an empty set is $+\infty$). Let $0 < \varepsilon < x < L$. Find a local martingale of the form $f(X_t)$ such that f is twice differentiable, $f(\varepsilon) = 1$ and $f(L) = 0$. Show that $M_t = f(X_{t\wedge\sigma_\varepsilon\wedge\sigma_L})$ is a bounded martingale.

(c) Consider $X_t - B_t$ and show that $\sigma_\varepsilon \wedge \sigma_L$ is almost surely finite. Find $\mathbb{P}^x(\sigma_\varepsilon < \sigma_L)$ by applying the optional stopping theorem to M_t.

(d) Use (c) to show that $\tau < \infty$ with positive probability if and only if $\delta < 2$. Show also that in that case, $\mathbb{P}(\tau < \infty) = 1$ and that $\lim_{t\nearrow\tau} X_t = 0$ almost surely.

Hint. For the last claim, you might need the strong Markov property of diffusions: if τ is an almost surely finite stopping time, then $Y_t = X_{\tau+t}$ is the solution of the same SDE with the Brownian motion $W_t = B_{\tau+t} - B_\tau$ and with the initial value $Y_0 = X_\tau$.

5. Girsanov transformation for random walk

Let $\Omega = \{\omega : \mathbb{Z}_+ \to \mathbb{Z} : \omega(0) = 0\}$. Let $X_n(\omega) = \omega(n)$, $n \in \mathbb{Z}_+$, be the coordinate maps and let $\mathcal{F}$ be the σ-algebra generated by X_n, $n \in \mathbb{Z}_+$. The space $(\Omega, \mathcal{F})$ is the canonical space for $\mathbb{Z}$-valued discrete-time stochastic processes. Suppose that there is defined a family of probability measures $\mathbb{P}_p$, $0 < p < 1$, on $(\Omega, \mathcal{F})$ such that for any $\mathbb{P}_p$ the increments $\xi_n = X_n - X_{n-1}$, $n \in \mathbb{N}$, are independent and identically distributed as

$$\mathbb{P}_p(\xi_n = 1) = p, \qquad \mathbb{P}_p(\xi_n = -1) = 1 - p.$$

Define a filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n) = \sigma(\xi_1, \ldots, \xi_n) \subset \mathcal{F}$.

(a) Let $\mathbb{P}|_\mathcal{A}$ be the restriction of a probability measure $\mathbb{P}$ on a σ-algebra $\mathcal{A}$. Find an explicit formula for the Radon-Nikodym derivative

$$M_n = \frac{\mathrm{d}\mathbb{P}_p|_{\mathcal{F}_n}}{\mathrm{d}\mathbb{P}_{1/2}|_{\mathcal{F}_n}}.$$

(b) Show that M_n is a martingale for the probability measure $\mathbb{P}_{1/2}$ and for the filtration $(\mathcal{F}_n)_{t\in\mathbb{Z}_+}$.

(c) Let $c \in \mathbb{R}$. For each $t \in \mathbb{R}_+$ and $0 < \varepsilon < |c|^{-1}$, define

$$n^{(\varepsilon)}(t) = \lfloor \varepsilon^{-2} t \rfloor, \qquad Y_t^{(\varepsilon)} = \varepsilon X_{n^{(\varepsilon)}(t)}, \qquad p^{(\varepsilon)} = \frac{1}{2}(1 + c\varepsilon)$$

where $\lfloor x \rfloor$ is the largest integer less or equal to the real number x. Find the (scaling) limit of $M_t^{(\varepsilon)} = M_{n^{(\varepsilon)}(t)}$ as $\varepsilon \searrow 0$ in terms of $t, Y_t = \lim Y_t^{(\varepsilon)}$ and c.