UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 7 (Nov 1)

1. Koebe distortion and growth theorem

(a) Remember that the class S consists of univalent functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$. Define for any fixed $f \in S$ and $w \in \mathbb{D}$, a function

$$h(z) = \frac{f\left(\frac{z+w}{1+\overline{w}\,z}\right) - f(w)}{(1-|w|^2)\,f'(w)}.$$

Show that $h \in S$ and that

$$h(z) = z + \left(\frac{1}{2}(1-|w|^2)\frac{f''(w)}{f'(w)} - \overline{w}\right) z^2 + \ldots$$

Show (using the special case of the Bieberbach–de Branges-theorem given in the lecture notes) that

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2}. \tag{1}$$

(b) Use the inequality (1) on the radial line $z = x \in (0,1)$ to show that

$$\frac{1-x}{(1+x)^3} \leq |f'(x)| \leq \frac{1+x}{(1-x)^3}.$$

(c) Use the previous inequality to show that

$$\frac{x}{(1+x)^2} \leq |f(x)| \leq \frac{x}{(1-x)^2}.$$

2. Let A be the set of all holomorphic $g : D_g \to \mathbb{C}$ such that the domain D_g of g contains a neighborhood of ∞ and that g has near ∞ an expansion of the form

$$g(z) = z + \sum_{k=1}^{\infty} a_k(g)\, z^{-k}$$

Let $g, \hat{g} \in A$.

(a) Show that $g \circ \hat{g} \in A$ and that $a_1(g \circ \hat{g}) = a_1(g) + a_1(\hat{g})$.

(b) Show that $g_\lambda(z) = \lambda g(z\lambda^{-1})$, $\lambda > 0$, is in A and that $a_1(g_\lambda) = \lambda^2\, a_1(g)$.

(c) Show that $g_{(x)}(z) = x + g(z-x)$ is in A and that $a_1(g_{(x)}) = a_1(g)$.

(d) It follows from the form of the expansion that $g \in A$ is injective in a neighborhood of ∞. Find the inverse $f = g^{-1}$ which has an expansion

$$f(z) = b_{-1}z + b_0 + b_1 z^{-1} + \ldots$$

near ∞ by giving a (semi-explicit) recursion for the coefficients b_k and calculate explicitly $b_{-1}, b_0, \ldots, b_3$ in terms of $a_k(g)$'s.

Turn page!

3. Poisson kernel in $\mathbb{H} \setminus \overline{B(0,R)}$

(a) Let $R > 0$ and let $H_R = \mathbb{H} \setminus \overline{B(0,R)}$. Show that any continuous bounded function $u : \overline{H_R} \to \mathbb{R}$ which is harmonic in H_R and satisfies $u(x) = 0$, $x \in \mathbb{R} \setminus (-R, R)$, can be represented as an integral

$$u(z) = \frac{2R}{\pi} \int_0^\pi \frac{\operatorname{Im}\left(z + R^2 z^{-1}\right) \sin\theta}{\left|z + R^2 z^{-1} - 2R\cos\theta\right|^2} \, u(Re^{i\theta}) \, \mathrm{d}\theta$$

for any $z \in H_R$. *Hint.* Make a conformal transformation to $\mathbb{H}$ and use the Poisson kernel of $\mathbb{H}$.

(b) Show that

$$\lim_{y \nearrow \infty} y \, u(iy) = \frac{2R}{\pi} \int_0^\pi u(Re^{i\theta}) \sin\theta \, \mathrm{d}\theta$$