UNIVERSITY OF HELSINKI

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Schramm–Loewner evolution, Fall 2011
Problem Sheet 6 (Oct 18)

1. Exponential Brownian motion

Let $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$, and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Solve the stochastic differential equation

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \qquad X_0 = x_0 > 0$$

by trying a solution of the form $X_t = f(t, B_t)$ where f is smooth enough function on $\mathbb{R}_+ \times \mathbb{R}$.

Note. There is a theorem in the lecture notes which gives sufficient conditions guaranteeing the solution exists and is unique.

2. Orstein–Uhlenbeck process

Let $\alpha, \sigma \in \mathbb{R}$ be positive and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Solve the stochastic differential equation

$$\mathrm{d}X_t = -\alpha X_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t, \qquad X_0 = x_0 \in \mathbb{R}$$

by trying a solution of the form $X_t = a(t)\big(x_0 + \int_0^t b(s)\,\mathrm{d}B_s\big)$ where a and b are smooth enough functions on $\mathbb{R}_+$. This form of the solution is motivated by the fact that we expect a Gaussian solution (see also Exercise 2 of Problem Sheet 5).

3. Let $x_0, x_1 \in \mathbb{R}$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Find the solution of

$$\mathrm{d}X_t = \frac{x_1 - X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t, \quad t \in [0,1), \qquad X_0 = x_0$$

by slightly adapting the guess solution of the previous exercise. Find the mean $\mathbb{E}(X_t)$ and the covariance $\mathbb{E}[(X_s - \mathbb{E}(X_s))(X_t - \mathbb{E}(X_t))]$ of this process.

4. If $A \in \mathbb{C}^{2\times 2}$, $\det A \neq 0$, define a Möbius map by

$$\phi_A(z) = \frac{a_{11}\,z + a_{12}}{a_{21}\,z + a_{22}}$$

where $A_{ij} = a_{ij}$. Show that

$$\phi_A \circ \phi_B = \phi_{AB}\ .$$

Use this to find the inverse map of any Möbius map.

Turn page!

5. The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the spherical metric

Consider the complex plane as a subset of $\mathbb{R}^3$ by associating $\mathbb{C}$ with the the subspace of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 = 0$. A standard construction of the extended complex plane is through the *stereographic projection*: each point P on the sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\} \subset \mathbb{R}^3$ other than $N = (0, 0, 1) \in S$ is projected to the complex plane by the taking the line going through P and N and finding the unique intersection point z_P of this line and the complex plane. As the point P approaches N, $|z_P|$ goes to infinity. This defines a mapping from the sphere S to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(a) For each $z \in \mathbb{C} \subset \mathbb{R}^3$, find $v_z \in S$ of the form

$$v_z = \lambda_z \, z + (1 - \lambda_z) N$$

where $0 < \lambda_z \leq 1$. Show that the map $z \mapsto v_z$ defines a smooth map from $\mathbb{C}$ onto $S \setminus \{N\}$ and that $v_z \to N$ as $|z| \to \infty$.

(b) Show that the map $z \mapsto v_z$ is conformal in the sense that the vectors $\partial_x v_z$ and $\partial_y v_z$ are orthogonal and have the same length. Here the partial derivatives are with respect to the real and imaginary parts of z.

(c) Find the *spherical metric* $\rho : \hat{C} \times \hat{C} \to \mathbb{R}$ defined by

$$\rho(z, w) = |v_z - v_w|, \; z, w \in \mathbb{C}, \qquad \rho(z, \infty) = |v_z - N|, \; z \in \mathbb{C}, \qquad (\, \rho(\infty, \infty) = 0 \,)$$

where $| \cdot |$ is the Euclidian norm in $\mathbb{R}^3$. Show that the spherical metric is invariant under any map ϕ_A (as in Exercise 4) where $A \in \mathbb{C}^{2\times 2}$ is unitary, that is, $AA^* = I$ where A^* is the adjoint (conjugate transpose) of A.

Hint. For each $z \in \mathbb{C}$, define

$$\tilde{z} = \begin{bmatrix} z \\ 1 \end{bmatrix} \in \mathbb{C}^2$$

and notice that

$$\widetilde{\phi_A(z)} = \frac{1}{a_{21} z + a_{22}} A\tilde{z}.$$

What are the norms of $\tilde{z}$ and $\tilde{z} - \tilde{w}$?