UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 5 (Oct 11)

1. We call a expression of the form

$$\mathrm{d}X_t = m_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t \tag{1}$$

the *Itô differential* of X_t. Let $r, q \in \mathbb{N}$. Find the Itô differentials of the following processes:

(a) $X_t = B_t^q$, (b) $X_t = (\sin B_t)^r$, (c) $X_t = B_t^q\,(\sin B_t)^r$.

2. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-random square-integrable function. Show that

$$X_t = \int_0^t f(s)\,\mathrm{d}B_s$$

is normally distributed. Find the mean and the variance of X_t.

3. Integration by parts

(a) For for any semimartingales X_t and Y_t, show that the following integration by parts formula holds

$$\int_0^t X_s\,\mathrm{d}Y_s = X_t\,Y_t - X_0\,Y_0 - \int_0^t Y_s\,\mathrm{d}X_s - \langle X, Y\rangle_t.$$

Note. As implicitly defined in Itô's formula for semimartingales in the lecture notes, here the definition for the integrals $\int_0^t X_s\,\mathrm{d}Y_s$ and $\int_0^t Y_s\,\mathrm{d}X_s$ is such that

$$\int_0^t f_s\,\mathrm{d}X_s = \int_0^t f_s\,m_s\,\mathrm{d}s + \int_0^t f_s\,\sigma_s\,\mathrm{d}B_s \tag{2}$$

for the semimartingale of the form (1) whenever the two integrals on the right of (2) make sense. For more general semimartingales extend accordingly.

(b) Use Itô's formula for $B_t f(t)$ and $F(B_t)$, where f and F are smooth enough non-random functions, to find two integration by parts formulas for the integrals of types

$$\int_0^t f(s)\mathrm{d}B_s, \qquad \int_0^t f(B_s)\mathrm{d}B_s.$$

Note that this gives a way interpret these integrals in pathwise (ω-by-ω) sense. Why?

4. Let $\theta \in \mathbb{R}$ and $X_t = \exp\left(\theta B_t - \frac{\theta^2}{2} t\right)$. Show using Itô's formula that $(X_t)_{t\in\mathbb{R}_+}$ is a local martingale. Is it also a martingale?

5. (a) Let $(B_t)_{t\in\mathbb{R}_+}$, $B_t = (B_t^{(1)}, \ldots, B_t^{(m)})$, be m-dimensional standard Brownian motion, $m \geq 2$, started from $B_0 \neq 0$. The Euclidian norm in $\mathbb{R}^m$ is denoted by $|\cdot|$. Show using Itô's formula that $Y_t = |B_t|$ satisfies

$$\mathrm{d}Y_t = \sum_k \frac{B_t^{(k)}\mathrm{d}B_t^{(k)}}{Y_t} + \frac{m-1}{2Y_t}\mathrm{d}t$$

(b) Calculate $\langle Y\rangle_t$.

Note. You can use the following information about m-dimensional standard Brownian motion send away from the origin: when $m \geq 2$, almost surely $Y_t = |B_t| > 0$ for all $t \in \mathbb{R}_+$, i.e. almost surely the Brownian motion doesn't hit the origin. Why is this important?