UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 4 (Oct 4)

1. Modulus of continuity of Brownian motion

(a) This exercise continues the exercises 4 and 5 of Problem Sheet 2. Remember that there we made a construction of a Brownian motion on $[0,1]$ as a uniformly convergent series

$$B_t = \sum_{n=0}^{\infty} Z_t^{(n)}$$

where $(Z_t^{(n)})_{t\in[0,1]}$ are some piecewise linear processes. Remember also that for each $c > \sqrt{2\log 2}$ there exist an almost surely finite random variable N such that for $n \geq N$,

$$\|Z^{(n)}\|_\infty \leq c\, 2^{-\frac{n+1}{2}} \sqrt{n}.$$

Show that for any n and $h > 0$ and $0 \leq t \leq 1-h$

$$\left| Z_{t+h}^{(n)} - Z_t^{(n)} \right| \leq \min\{h \cdot 2^n, 2\}\, \|Z^{(n)}\|_\infty$$

and use this bound to show that there exists a constant $C > 0$ and a random variable $\Delta > 0$ such that for any $0 < h < \Delta$,

$$|B_{t+h} - B_t| \leq C\sqrt{h \log(1/h)}.$$

Hint. For the last claim, divide the sum $\sum_{n=0}^{\infty} \left| Z_{t+h}^{(n)} - Z_t^{(n)} \right|$ as $\sum_{n=0}^{N-1} + \sum_{n=N}^{l} + \sum_{n=l+1}^{\infty}$ where N is as above. Then choose $\Delta > 0$ such that the first sum is less than $\sqrt{h\log(1/h)}$ and after that optimize over l.

(b) Show that for each $0 < \alpha < 1/2$ there exists an almost surely finite random variable $C_\alpha > 0$ such that for any $0 \leq s, t \leq 1$,

$$|B_t - B_s| \leq C_\alpha |t-s|^\alpha.$$

2. Doob's maximal inequality in discrete time

Let $(M_n)_{n\in\mathbb{Z}_+}$ be a non-negative discrete-time submartingale with respect to a (discrete-time) filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$.

(a) Show that

$$\mathbb{E}[M_m \mathbb{1}_A] \leq \mathbb{E}[M_n \mathbb{1}_A]$$

for all $A \in \mathcal{F}_m$ and all $n \geq m$.

(b) Let $\tau = \min\{m : M_m \geq \lambda\}$. Show that

$$\lambda \mathbb{1}_{\tau \leq n} \leq \sum_{m=0}^{n} M_m \mathbb{1}_{\{\tau = m\}}$$

(c) Show that

$$\mathbb{P}\left(\max_{0\leq m\leq n} M_m \geq \lambda \right) \leq \frac{1}{\lambda}\, \mathbb{E}(M_n)$$

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3. Doob's maximal inequality in continuous time

Let $(M_t)_{t\in\mathbb{R}_+}$ be a continuous non-negative (continuous-time) submartingale with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Use the previous exercise to show that for any $T > 0$

$$\mathbb{P}\left(\sup_{0\leq t\leq T} M_t \geq \lambda\right) \leq \frac{1}{\lambda}\,\mathbb{E}(M_T).$$

Hint. Set $\pi_n = \{k\,T\,2^{-n} \,:\, k = 0, 1, 2, \ldots, 2^n\}$ for any $n \in \mathbb{N}$ and consider first $\sup_{t\in\pi_n} M_t$.

4. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Show that B_t and $B_t^2 - t$ are martingales with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

5. Let $f \in \mathcal{L}^2$ be simple and bounded and define $X_t(\omega) = \int_0^t f(s,\omega)\mathrm{d}B_s(\omega)$ and $\langle X\rangle_t(\omega) = \int_0^t f(s,\omega)^2\mathrm{d}s$. Show that $X_t^2 - \langle X\rangle_t$ is a martingale. Show also that $\langle X\rangle$ is the quadratic variation process $V_X^{(2)}$ in the sense of the definition in Section 1.3 in the lecture notes.