UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 3 (Sep 27)

1. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Show that $(B_t)_{t\in\mathbb{R}_+}$ satisfies *Brownian scaling*: if $r>0$ then the process $(X_t)_{t\in\mathbb{R}_+}$ defined by $X_t = r^{-1/2}B_{rt}$ is a standard one-dimensional Brownian motion.

2. Show also that the process $(Y_t)_{t\in\mathbb{R}}$ defined by

$$Y_t = \begin{cases} 0 & \text{when } t = 0 \\ tB_{1/t} & \text{when } t > 0 \end{cases}$$

is a standard one-dimensional Brownian motion.

Hint. The following result about Gaussian random variables might be useful: If the random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian (also called multivariate normal), then they are independent if and only if $\mathbb{E}((X_j - \mathbb{E}X_j)(X_k - \mathbb{E}X_k)) = 0$ for any $j \neq k$.

3. A *standard d-dimensional Brownian motion* is an $\mathbb{R}^d$-valued stochastic process $(B_t^{(1)}, \ldots, B_t^{(d)})$ where $B_t^{(1)}, \ldots, B_t^{(d)}$ are independent standard one-dimensional Brownian motions.

Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard d-dimensional Brownian motion and let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal transformation (a linear mapping $x \mapsto A\,x$ with $A^T = A^{-1}$). Show that the process $(Z_t)_{t\in\mathbb{R}_+}$ defined by $Z_t = A\,B_t$ is a standard d-dimensional Brownian motion.

4. Let X_n, $n\in\mathbb{N}$, and X be random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the following statements are equivalent:

(1) $X_n \to X$ almost surely, i.e. $\mathbb{P}(\{\omega : \lim_n X_n(\omega) = X(\omega)\}) = 1$.

(2) $\lim_{m\to\infty} \mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| < \varepsilon \text{ for all } n \geq m) = 1$ for any $\varepsilon > 0$.

(3) $\lim_{m\to\infty} \mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon \text{ for some } n \geq m) = 0$ for any $\varepsilon > 0$.

5. Remember that we say that $X_n \to X$ in probability if and only if for each $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$.

(a) Show that if $X_n \to X$ almost surely, then $X_n \to X$ in probability.

(b) Show that if $X_n \to X$ in L^p, then $X_n \to X$ in probability. Show also that if $X_n \to X$ in probability and $|X_n| \leq Y$ for some non-negative random variable $Y \in L^p$, then $X_n \to X$ in L^p.

(c) Give an example of a sequence of random variables X_n which converges in L^p, but not almost surely.

(d) Show that if $X_n \to X$ in probability, then there exist a subsequence X_{n_j} such that $X_{n_j} \to X$ almost surely.