UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 2 (Sep 20)

1. Borel-Cantelli lemma

Let A_k, $k \in \mathbb{N}$ be a sequence of events. Define $\{\omega : \omega \in A_k \text{ i.o.}\}$, where i.o. stands for *infinitely often*, as the event $\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j$. Show that if $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$ then

$$\mathbb{P}(A_n \text{ i.o.}) = 0.$$

Hint. Consider the random variable $N = \sum_{k=1}^{\infty} \mathbb{1}_{A_k}$.

2. Brownian bridge

(a) Let $(B_t)_{t \in \mathbb{R}_+}$ be a standard one-dimensional Brownian motion. For any $0 < t < s$, find the conditional density of B_t given B_s in the sense of Exercise 5 of Problem Sheet 1. Illustrate this distribution by drawing its mean and standard deviation as functions of t.

(b) Let $0 \leq t < s$. Show that for any $\lambda \in (0, 1)$, the random variable

$$Y_\lambda = B_{\lambda t + (1-\lambda)s} - \lambda B_t - (1 - \lambda) B_s$$

is independent from $\sigma(B_u\,,\, 0 \leq u \leq t)$ and $\sigma(B_u\,,\, u \geq s)$.

(c) Conclude from (a)–(b) that for any $0 \leq r < s$ and $x, y \in \mathbb{R}$, conditionally on $B_r = x$ and $B_s = y$ the processes $(B_t)_{t \in [0,r)}$, $(B_t)_{t \in (r,s)}$ and $(B_t)_{t \in (s,\infty)}$ are independent. Furthermore, if $0 = s_0 < s_1 < \ldots < s_n$ and $x_k \in \mathbb{R}$ and $t_k \in (s_{k-1}, s_k)$ for $k = 1, 2, \ldots, n$, then describe the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ given $B_{s_1} = x_1, B_{s_2} = x_2, \ldots, B_{s_n} = x_n$.

3. (a) Let X be a Gaussian random variable with mean 0 and variance 1. Show that for any $x > 0$,

$$\mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}\, x} \exp\left(-\frac{1}{2}x^2\right).$$

(b) Let $X_n \sim N(\mu_n, \sigma_n^2)$ be a sequence of Gaussian random variables such that $X_n \to X$ almost surely and $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ as $n \to \infty$. Show that $X \sim N(\mu, \sigma^2)$. (Here, as usual, $X \sim N(\mu, \sigma^2)$ means that X is distributed normally with mean μ and variance σ^2.)

Turn page!

4. A construction of Brownian motion on $[0,1]$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a countably infinite set $(\xi(t))_{t\in\mathcal{D}}$ of independent $N(0,1)$ random variables, where $\mathcal{D} = \bigcup \mathcal{D}_n$ and $\mathcal{D}_n = \{k\,2^{-n} : k = 0,1,2,\dots,2^n\}$ are the sets of *dyadic* rationals. (Such probability space can be constructed, for example, as a product space.)

(a) We are first going to construct a process $B(t)$, $t \in \mathcal{D}$, such that it agrees with the distribution of Brownian motion on the dyadic points.

On $\mathcal{D}_0 = \{0,1\}$, define $B(0) = 0$ and $B(1) = c_0\xi(1)$. Define then recursively $B(t)$ on $t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ for $n \geq 1$, by setting

$$B(t) = a_n\, B(t-2^{-n}) + b_n\, B(t+2^{-n}) + c_n\,\xi(t)$$

for any $t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Find the real numbers c_0, a_n, b_n, c_n, $n \geq 1$ such that $B(1)$ has the correct distribution (Brownian motion at time 1) and the conditional distribution of $B(t), t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$, given $B(t), t \in \mathcal{D}_{n-1}$, is the correct one (the Brownian bridge distribution from Exercise 3(c)).

(b) Now we extend the process $B(t)$ to the whole interval $[0,1]$.

Let $B|_{\mathcal{D}_n}$ be the restriction of B to $\mathcal{D}_n$ and define $B^{(n)}(t)$ as the extension of $B|_{\mathcal{D}_n}$ by linear interpolation to $[0,1]$. Define piecewice linear processes $Z^{(0)} = B^{(0)}$ and $Z^{(n)} = B^{(n)} - B^{(n-1)}$, $n \geq 1$. Show that

$$\sum_{n=0}^{\infty} \mathbb{P}(\text{for some } t \in \mathcal{D}_n,\ |\xi(t)| \geq c\sqrt{n}) < \infty$$

when $c > \sqrt{2\log 2}$. Use Borel–Cantelli lemma (Exercise 1) to conclude that the sum $\sum_{n=0}^{\infty} \|Z^{(n)}\|_\infty$ is almost surely finite and that almost surely the series

$$B(t) = \sum_{n=0}^{\infty} Z^{(n)}(t)$$

converges uniformly in $[0,1]$.

5. Let $(B_t)_{t\in[0,1]}$ be the process constructed in the previous exercise. Prove that the increments of B_t are independent and that $B_{s+t} - B_s$ is normally distributed with mean 0 and variance t.