UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 12 (Dec 13)

1. Schwarzian derivative

Define the Schwarzian derivative of f at z as

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

for any function f which is locally conformal near z, that is, f is holomorphic in a neighborhood of z with $f'(z) \neq 0$.

(a) Show that S satisfies

$$(S(f \circ g))(z) = g'(z)^2 Sf(g(z)) + Sg(z)$$

for any functions f and g that are locally conformal near $g(z)$ and z, respectively.

(b) Show that S satisfies

$$(S(\phi \circ f \circ \psi))(z) = \psi'(z)^2 Sf(\psi(z))$$

for all Möbius maps ϕ and ψ.

(c) Let A be a hull with $0 \notin A$ and define $\Phi_A(z) = g_A(z) - g_A(0)$. Show that

$$a_1(\tilde{A}) = -\frac{1}{6} S\Phi_A(0)$$

where $a_1(\tilde{A})$ is the half-plane capacity of the hull $\tilde{A} = \{-z^{-1} \, : \, z \in A\}$. Deduce that $S\Phi_A(0) < 0$ unless Φ_A is a identity map.

2. Show using the Koebe distortion theorem that there exists constants C and r such that for any conformal map $f : \mathbb{H} \to \mathbb{C}$ and for any $x \in \mathbb{R}$, $y > 0$ and $1/2 \leq s \leq 2$

$$C^{-1}|f'(iy)| \leq |f'(isy)| \leq C|f'(iy)|$$
$$C^{-1}(1+x^2)^{-r}|f'(iy)| \leq |f'(y(x+i))| \leq C(1+x^2)^r|f'(iy)|.$$

What is the value of r that you get from the Koebe distortion theorem?

3. (a) Let g_t be a Loewner chain and $f_t = g_t^{-1}$. By differentiating the Loewner equation of f_t with respect to z, find a differential equation for $f_t'(z)$. Show that for $x \in \mathbb{R}$, $y > 0$

$$|\partial_t f_t'(x+iy)| \leq \frac{2|f_t''(x+iy)|}{y} + \frac{2|f_t'(x+iy)|}{y^2}.$$

(b) Show using the special case $|a_2| \leq 2$ of the Bieberbach–de Branges theorem that there is a constant $c > 0$ such that

$$|f''(z)| \leq \frac{c}{\operatorname{Im} z}|f'(z)|$$

for any $f : \mathbb{H} \to \mathbb{C}$ conformal and for any $z \in \mathbb{H}$.

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(c) Show that there are constants c_1, c_2, c_3 such that following holds for any Loewner chain: for any $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y > 0$

$$|\partial_t f_t'(x+iy)| \leq \frac{c_1 |f_t'(x+iy)|}{y^2}$$

and if $0 \leq s \leq y^2$ then

$$|f_{t+s}'(x+iy)| \leq c_2\,|f_t'(x+iy)|$$
$$|f_{t+s}(x+iy) - f_t(x+iy)| \leq c_3\, y^2\,|f_t'(x+iy)|.$$

4. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion and let $\alpha \in \mathbb{R}$. Define $(X_t)_{t\in\mathbb{R}_+}$ by $X_t = B_t + \alpha t$.

(a) Let $0 = t_0 < t_1 < \ldots < t_n = T$. Write the joint probability density of X_t, $t \in \{t_0, t_1, \ldots, t_n\}$, with respect to the joint probability density of B_t, $t \in \{t_0, t_1, \ldots, t_n\}$.

(b) Let $T > 0$. Show that the laws of $(B_t)_{t\in[0,T]}$ and $(X_t)_{t\in[0,T]}$ are absolutely continuous with respect to each other. Define $(Y_t)_{t\in\mathbb{R}_+}$ by $Y_t = \sqrt{\kappa} B_t$, $\kappa \geq 0$. Show that the laws of $(B_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are mutually singular unless $\kappa = 1$. Remember that probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually singular if there is an event E such that $\mathbb{P}(E) = 1$ and $\mathbb{Q}(E) = 0$.

Hint. For the second claim, consider the quadratic variation of these processes.

(c) Show that the laws of $(B_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ are mutually singular when $\alpha \neq 0$.

Hint. When $\alpha > 0$, consider the probabilities $\mathbb{P}(X_t \to \infty \text{ as } t \to \infty)$ and $\mathbb{P}(B_t \to \infty \text{ as } t \to \infty)$.