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Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 11 (Nov 29)

1. Let f be holomorphic and $Z_t = X_t + iY_t$ be a complex semimartingale, i.e. the real and imaginary parts X_t and Y_t are semimartingales.

(a) Show that if $\langle Y \rangle_t = 0$ for all t, then

$$\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t + \frac{1}{2}f''(Z_t)\mathrm{d}\langle X \rangle_t.$$

(b) Show that if $\langle X \rangle_t = \langle Y \rangle_t$ and $\langle X, Y \rangle_t = 0$ for all t, then

$$\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t.$$

(c) In the general case, find an expression for $\langle Z \rangle_t$ so that Itô's formula can be written as

$$\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t + \frac{1}{2}f''(Z_t)\mathrm{d}\langle Z \rangle_t.$$

2. Let's deal with both the forward and reverse Schramm–Loewner evolution by fixing $\nu = \pm 1$ and letting $g_t(z)$ be the solution of the following equation

$$\partial_t g_t(z) = \nu \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z$$

where $W_t = -\sqrt{\kappa} B_t$. Let $Z_t = g_t(z) - W_t$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively. Verify all the following formulas

$$\mathrm{d}X_t = 2\nu \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \partial_t Y_t = -2\nu \frac{Y_t}{X_t^2 + Y_t^2}, \qquad \partial_t \frac{|g_t'(z)|}{Y_t} = 4\nu \frac{|g_t'(z)|}{Y_t} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2}$$

$$\mathrm{d} \arg Z_t = (\kappa - 4\nu) \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{Y_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,$$

$$\mathrm{d} \log |Z_t| = -\frac{1}{2}(\kappa - 4\nu) \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,$$

$$\mathrm{d} \sin \arg Z_t = (\sin \arg Z_t) \left[\frac{(\kappa - 4\nu)X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t \right]$$

3. Let $U \subset \mathbb{C}$ be a simply connected domain with $U \neq \mathbb{C}$ and let $z_0 \in U$. Let ψ be the unique conformal map from U onto $\mathbb{D}$ such that $\psi(z_0) = 0$ and $\psi'(z_0) > 0$. Then the *conformal radius of U from z_0* is defined as

$$\rho(z_0, U) = \frac{1}{\psi'(z_0)}.$$

(a) Show that if ϕ is a conformal map from U onto $\mathbb{D}$ with $\phi(z_0) = 0$ then $\rho(z_0, U) = |\phi'(z_0)|^{-1}$. Show also that $\rho(\lambda z_0, \lambda U) = \lambda \rho(z_0, U)$ for $\lambda > 0$ and $\rho(f(z_0), f(U)) = |f'(z_0)| \rho(z_0, U)$ for any conformal map $f : U \to \mathbb{C}$.

(b) Show using the Koebe distortion theorem that

$$\mathrm{dist}(z_0, \partial U) \leq \rho(z_0, U) \leq 4\, \mathrm{dist}(z_0, \partial U).$$

(c) Let g be a conformal map from U onto $\mathbb{H}$. Show that

$$\rho(z_0, U) = \frac{2 \operatorname{Im} g(z_0)}{|g'(z_0)|}.$$

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4. Dimension of SLE(κ) is $1+\frac{\kappa}{8}$

Consider SLE(κ). Let's assume that there is some $\mu > 0$ such that the limit

$$h(z) = \lim_{\varepsilon \searrow 0} \varepsilon^{-\mu} \, \mathbb{P}(\rho(z, U_z) \leq \varepsilon)$$

exists and h is smooth and positive in $\mathbb{H}$. Here U_z is the connected component of z in $\mathbb{H} \setminus \gamma[0,\infty)$.

(a) Show that $h(\lambda z) = \lambda^{-\mu} h(z)$. Write $h(x+i\,y) = y^{-\mu}\tilde{h}(x/y)$.

(b) Find a second order differential operator $\mathcal{D}$ such that $\mathcal{D}\tilde{h} = 0$. *Hint.* $\mathbb{P}(\rho(z, U_z) \leq \varepsilon \,|\, \mathcal{F}_t)$ is a martingale. You can assume that convergence in the definition of h is sufficiently uniform so that $\lim_{\varepsilon \searrow 0} \varepsilon^{-\mu} \, \mathbb{P}(\rho(z, U_z) \leq \varepsilon \,|\, \mathcal{F}_t)$ is a local martingale.

(c) Find a positive solution of $\mathcal{D}\tilde{h} = 0$ by trying a solution of the form $\tilde{h}(u) = (1+u^2)^\alpha$. What is the value of μ?

5. For $\kappa > 4$, almost surely $\tau(z) < \infty$

(a) Let $\nu \in \mathbb{C}$ with $|\nu| = 1$ and let $0 < \alpha < 1$. Show that $z \mapsto \nu z^\alpha$ defines a conformal conformal map from $\mathbb{H}$ to $\mathbb{C}$ (for any choice of the branch, if you wish). Find $\nu = \nu(\alpha)$ such that the image domain is symmetric with respect to the y-axis and lies in $\mathbb{H}$.

(b) Let $h(z) = \mathrm{Im}(\nu z^\alpha)$ where $\nu = \nu(\alpha)$ is as above. Show that there is a constant $C = C(\alpha) \geq 1$ such that $C^{-1}|z|^\alpha \leq h(z) \leq C|z|^\alpha$ for all $z \in \overline{\mathbb{H}}$.

(c) Consider SLE(κ) with $\kappa > 4$ and let $\alpha = \alpha(\kappa) = 1 - 4/\kappa$. Show that the real and imaginary parts of Z_t^α are local martingales where $Z_t = g_t(z) - W_t$, $z \in \mathbb{H}$, and $W_t = -\sqrt{\kappa}B_t$. Conclude that $h(Z_t)$ is a local martingale.

(d) For any $R > 0$, define $\sigma_R = \tau(z) \wedge \inf\{t \in [0,\tau(z)) : |Z_t| = R\}$. We make the assumption that $\sigma_R < \infty$ almost surely. (See Rohde&Schramm, Basic properties of SLE, Lemma 6.5., for the proof of this fact.) What is the geometric description of σ_R, that is, it is the exit time of Z_t from which set? Show that $h(Z_{t\wedge\sigma_R})$ is a martingale and show that there exist a constant $\tilde{C} = \tilde{C}(\kappa) \geq 1$ such that

$$\tilde{C}^{-1}\left(\frac{|z|}{R}\right)^\alpha \leq \mathbb{P}\left(|Z_{\sigma_R}| = R\right) \leq \tilde{C}\left(\frac{|z|}{R}\right)^\alpha.$$

Hint. Use the optional stopping theorem.

(e) Deduce that for $\kappa > 4$ and for any $z \in \mathbb{H}$, almost surely $\tau(z) < \infty$.