UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 10 (Nov 22)

1. Define $m(z) = -\overline{z}$ which is an injective antiholomorphic self-map of $\mathbb{H}$. Show that SLE(κ), $\kappa > 0$, is *symmetric*, i.e., $(m(K_t))_{t\in\mathbb{R}_+}$ and $(K_t)_{t\in\mathbb{R}_+}$ are equal in distribution. Is the random Loewner chain with the driving process $W_t = \sqrt{\kappa}B_t + \alpha t$ symmetric?

2. Time change of a semimartingale

(a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let $a(t,\omega)$ be a continuous, positive, adepted process. Define a random time-change by setting:

$$S(t,\omega) = \int_0^t a(r,\omega)^2 \,\mathrm{d}r, \qquad \sigma(s,\omega) = \inf\{t \in \mathbb{R}_+ \,:\, S(t,\omega) \geq s\}$$

Let

$$\tilde{B}_s(\omega) = \int_0^{\sigma(s)} a(r,\omega)\,\mathrm{d}B_r(\omega),$$

By Theorem 1.6.8, $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$ and therefore we know how to construct Itô integral with respect to $\tilde{B}_s$. Show that the following time-change formula holds: for continuous, adapted process $v(t,\omega)$

$$\int_0^s v(\sigma(q),\omega)\,\mathrm{d}\tilde{B}_q(\omega) = \int_0^{\sigma(s)} v(r,\omega)a(r,\omega)\,\mathrm{d}B_r(\omega)$$

Hint. Check this first for $v(r,\omega) = \mathbb{1}_{[\sigma(s_1),\sigma(s_2))}(r)$.

(b) Let X_t be a semimartingale

$$\mathrm{d}X_t(\omega) = u(t,\omega)\mathrm{d}t + v(t,\omega)\mathrm{d}B_t(\omega).$$

Show that the process $(\tilde{X}_s)_{s\in\mathbb{R}_+}$ defined by

$$\tilde{X}_s = X_{\sigma(s)}$$

is a semimartingale with respect to $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$ and $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ and satisfies

$$\mathrm{d}\tilde{X}_s = \frac{u(\sigma(s))}{a(\sigma(s))^2}\mathrm{d}s + \frac{v(\sigma(s))}{a(\sigma(s))}\mathrm{d}\tilde{B}_s.$$

3. (a) Consider a strip $S_\pi = \{z \in \mathbb{C} \,:\, 0 < \operatorname{Im} z < \pi\}$. Show that

$$\phi_c : z \mapsto \log(z - c)$$

is a conformal map from $\mathbb{H}$ onto S_π. Show that the image of 0 is in $\mathbb{R}$ if and only if $c < 0$.

(b) Let $K \subset \overline{\mathbb{H}}$ be a hull. Let $x \in \mathbb{R} \setminus K$ and $x' = g_K(x)$. Show that

$$G = \phi_{x'} \circ g_K \circ \phi_x^{-1}$$

satisfies

$$G(z) = z - 2a_0 + o(1),\ z \to -\infty, \qquad G(z) = z + o(1),\ z \to +\infty.$$

Find a_0 in terms of g_K and x. Is a_0 positive?

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(c) Show that the conformal onto maps $f : S_\pi \to S_\pi$ with $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$ are of the form

$$f(z) = z + C$$

where $C \in \mathbb{R}$ is a constant.

(d) Let's call a compact $K \subset \overline{S_\pi}$ a *s-hull* or a *hull in the strip* if $S_\pi \setminus K$ is simply connected. Show that for any s-hull K, there exists a unique conformal map G_K from $S_\pi \setminus K$ onto S_π such that

$$G_K(z) = z - a_0 + o(1),\ z \to -\infty, \qquad G_K(z) = z + a_0 + o(1),\ z \to +\infty \tag{1}$$

for some constant $a_0 \in \mathbb{R}$. The constant a_0 could be called the *s-capacity* or the *strip capasity* of K. Show that it is additive when mappings with this form of expansion are composed.

4. Chordal SLE(κ) in the strip S_π

(a) Let $c < 0$. Let $(g_t)_{t\in\mathbb{R}_+}$ be a Loewner chain (in $\mathbb{H}$) with the driving term $(W_t)_{t\in\mathbb{R}_+}$. Motivated by the previous exercise define a time-change $\sigma(s)$ (which depends on c and $(W_t)_{t\in\mathbb{R}_+}$) such that

$$-\frac{1}{2}\log g'_{\sigma(s)}(c) = s$$

and define

$$G_s(z) = \log\left[g_{\sigma(s)}\left(c + |c|e^z\right) - g_{\sigma(s)}\left(c\right)\right] + s - \log|c|.$$

Show that G_s is a conformal map from $S_\pi \setminus \tilde{K}_s$ onto S_π normalized as in (1) where $(\tilde{K}_S)_{s\in\mathbb{R}_+}$ is a growing family of s-hulls and $(G_s)_{s\in\mathbb{R}_+}$ satisfies the Loewner equation of S_π, i.e.,

$$\partial_s G_s(z) = \coth\frac{G_s(z) - \tilde{W}_s}{2}, \qquad G_0(z) = z \tag{2}$$

where $\tilde{W}_s = \log(W_{\sigma(s)} - g_{\sigma(s)}(c)) + s - \log|c|$.

(b) Show that when g_t is SLE(κ), that is, $W_t = \sqrt{\kappa}B_t$, then $\tilde{W}_s$ is a Brownian motion with drift.

5. Reverse the calculations of the previous problems: Suppose that G_s satisfies (2) with a driving process $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$ (note that we have general $\kappa \geq 0$ and $\alpha \in \mathbb{R}$ here). Find the conformal transformation from S_π to $\mathbb{H}$ and the correct time-change so that G_s is transformed to g_t which satisfies the Loewner equation of $\mathbb{H}$ with a driving process W_t. Is g_t well-defined for all t?

Hint. You might need a similar calculation as in Exercise 2 of Problem sheet 9 for G_s to get the correct normalization for g_t. Also, the full answer to this exercise should show that W_t satisfies the following system of stochastic differential equations:

$$\mathrm{d}W_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\rho}{W_t - C_t}\mathrm{d}t$$

$$\mathrm{d}C_t = \frac{2}{C_t - W_t}\mathrm{d}t$$

where $\rho = \rho(\kappa, \alpha) \in \mathbb{R}$ is a constant. Notice that the equation for $\mathrm{d}C_t$ is the Loewner equation of $\mathbb{H}$.