UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Schramm–Loewner evolution, Fall 2011
Problem Sheet 1 (Sep 13)

1. Let X be a non-negative random variable, and let h be continuously differentiable and non-decreasing function on $\mathbb{R}_+ = [0, \infty)$ such that $h(0) = 0$ (for example, $h(x) = x^p$, $p > 0$). Show that

$$\mathbb{E}(h(X)) = \int_0^\infty h'(y)\mathbb{P}(X \geq y)\mathrm{d}y.$$

2. Show that for any random variable $X \geq 0$ and any real number $x > 0$

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x}.$$

3. Let $\mathcal{A}$ be a σ-algebra such that for each $A \in \mathcal{A}$, $\mathbb{P}(A) = 0$ or 1. Show that $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}(X)$ for each $X \in L^1$.

4. Let $\Omega_1, \Omega_2, \ldots$ be a finite or countably infinite partition of Ω into $\mathcal{F}$-measurable sets, i.e., $\Omega_j \cap \Omega_k = \emptyset$ when $j \neq k$ and $\bigcup_{k=1}^\infty \Omega_k = \Omega$. Assume that each Ω_k has positive probability. Let $\mathcal{G}$ be the σ-algebra generated by $\Omega_1, \Omega_2, \ldots$ Show that

$$\mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}(X; \Omega_k)}{\mathbb{P}(\Omega_k)} \quad \text{on } \Omega_k.$$

Here we use the standard notation $\mathbb{E}(X; E) = \int_E X \mathrm{d}\mathbb{P}$.

5. Let X and Y be two random variables that have a joint density $f(x, y)$ in the sense that for any bounded, Borel (measurable) function ϕ on $\mathbb{R}^2$

$$\mathbb{E}(\phi(X, Y)) = \int_{\mathbb{R}^2} \phi(x, y) f(x, y) \,\mathrm{d}x \,\mathrm{d}y.$$

Define the marginal density of Y by

$$f_Y(y) = \int_{\mathbb{R}} f(x, y) \,\mathrm{d}x$$

and let

$$f(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & \text{, if } f_Y(y) > 0 \\ 0 & \text{, if } f_Y(y) = 0 \end{cases}$$

(a) Show that $f(x|y)f_Y(y) = f(x, y)$ for almost every (x, y) with respect to the Lebesgue measure on $\mathbb{R}^2$. *Hint:* Prove first that the Lebesgue measure of $\{x : f(x|y)f_Y(y) \neq f(x, y)\}$ is zero for each y and then use Fubini's theorem.

(b) Show that $f(x|y)$ can be seen as the conditional density of X given that $Y = y$ in the sense that

$$\mathbb{E}(h(X)|Y)(\omega) = \int_{\mathbb{R}} h(x) f(x|Y(\omega))\mathrm{d}x$$

for any bounded Borel function h on $\mathbb{R}$.