Chapter 4

Schramm–Loewner evolution

4.1 Definition of Schramm–Loewner evolution and some simple properties

4.1.1 Schramm's principle

In this section, we present *Schramm's principle* which is a small calculation which shows which kind of random Loewner chains have connection to statistical physics. We present this principle in a heuristical level, but with some additional definitions and conditions this could be made a theorem that states "Schramm–Loewner evolutions (as defined below) are the only random curves satisfying conformal invariance and the domain Markov property (as defined below)". We expect those two properties to be satisfied by scaling limits of random interfaces of statistical physics models at criticality. The crucial invention of Oded Schramm (1961–2008) was in his paper released in 1999 that random curves can be described using the Loewner equation with a random driving term. This enabled him to define Schramm–Loewner evolutions.

Assume that we are given a collection of probability measures $(\mathbb{P}^{U,a,b})$ indexed by the set all triplets (U,a,b) where U is any simply connected domain and $a \neq b$ are any two boundary points of U. Assume that $\mathbb{P}^{U,a,b}$ is the law of a random curve $\gamma : [0,\infty) \to \mathbb{C}$ (the parametrization is arbitrary) such that $\gamma([0,\infty)) \subset \overline{U}$ and $\gamma(0) = a$, $\gamma(\infty) = b$. We assume that the family $(\mathbb{P}^{(U,a,b)})$ satisfies the following properties:

- Let ϕ_* denote the pushforward defined by $\phi_* \mathbb{P} = \mathbb{P} \circ \phi^{-1}$. The family $(\mathbb{P}^{(U,a,b)})$ satisfies **conformal invariance** (CI):
$$\phi_* \mathbb{P}^{(U,a,b)} = \mathbb{P}^{(\phi(U),\phi(a),\phi(b))}$$
- Let $\mathcal{F}_t = \sigma(\gamma|_{[0,t]})$. The family $(\mathbb{P}^{(U,a,b)})$ satisfies **domain Markov property** (DMP):
$$\mathbb{P}^{(U,a,b)}(\gamma|_{[t,\infty)} \in B \,|\, \mathcal{F}_t) = \mathbb{P}^{(U \setminus \gamma([0,t]),\gamma(t),b)}(\gamma \in B)$$
for any measurable set B in the space of curves (in what ever way that space is defined. . .).

We also assume that we can describe the curve γ by the Loewner equation in the sense that $\mathbb{P}^{(\mathbb{H},0,\infty)}$ is supported on curves which satisfy Theorem 3.2.7.

We now investigate the consequences of these assumptions. The first observation is that we need to describe only one of the measures in the family. CI fixes the rest of them. Hence let's investigate $\mathbb{P}^{(\mathbb{H},0,\infty)}$. By Theorem 3.2.7 for each realization of γ there is a driving term $(W_t(\gamma))_{t\in\mathbb{R}_+}$ such that the corresponding conformal maps g_t satisfy the Loewner equation. Here we also make a reparametrization with the half-plane capacity. Let's call the random driving term $(W_t)_{t\in\mathbb{R}_+}$ as *driving process* of the random curve γ.

Fix some $t \in \mathbb{R}_+$. Define $\hat{\gamma}(s) = g_t(\gamma(t+s)) - W_t$, $s \in \mathbb{R}_t$. By CI and the DMP, $\hat{\gamma}$ is distributed as γ and independent of the realization of $\gamma|_{[0,t]}$. The conformal map associated to the hull $\hat{\gamma}([0,s])$ is

$$\hat{g}_s(z) = \tilde{g}_{t,s}(z + W_t) - W_t = g_{t+s} \circ g_t^{-1}(z + W_t) - W_t.$$

Now by differentiating this in s

$$\begin{aligned}\partial_s \hat{g}_s(z) &= (\partial_s g_{t+s})(g_t^{-1}(z+W_t)) \\ &= \frac{2}{g_{t+s}(g_t^{-1}(z+W_t)) - W_{t+s}} = \frac{2}{\hat{g}_s(z) - (W_{t+s} - W_t)}\end{aligned}$$

Hence

$$\hat{W}_s = W_{t+s} - W_t$$

is the driving process of $\hat{\gamma}$ and since $\hat{\gamma}$ is distributed as γ, $\hat{W}_s$ is independent of $\mathcal{F}_t$ and it is distributed as W_s. Hence the continuous stochastic process $(W_t)_{t\in\mathbb{R}_+}$ has *independent and stationary increments.* The next theorem shows that $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion with drift. After the proof of that theorem we'll show that actually there is no drift in our case.

Theorem 4.1.1. *If $(X_t)_{t\in\mathbb{R}_+}$, $X_0 = 0$, is a continuous stochastic process which has independent and stationary increments, then there exists a standard one-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and real numbers $\kappa \geq 0$ and α such that $X_t = \sqrt{\kappa} B_t + \alpha t$.*

Remark. The process of the form $X_t = \sqrt{\kappa} B_t + \alpha t$ is called *Brownian motion with drift.* Since the law of Brownian motion is invariant in the reflection $B_t \to -B_t$, we can choose the constant in front of the Brownian motion non-negative. The above parametrization is such that $\mathbb{E}\, X_t = \alpha t$ and $\operatorname{Var} X_t = \kappa t$.

Remark. A counterexample to this theorem is a Poisson process. It has independent and stationary increments, but it is not continuous.

Remark. We do not assume a priori that X_t has finite moments, otherwise this is a simple application of a central limit theorem.

Sketch of proof. For more details, see Kallenberg's book [7] Theorems 5.15 and 13.4. It is enough to prove that X_t is Gaussian for any t. The rest then follows from linearity of the expected value and the variance. Also without loss of generality, we can restrict to $t = 1$ and prove that X_1 is Gaussian.

For each $n \in \mathbb{N}$, write $X_1 = \sum_{j=1}^{n} \xi_{n,j}$ where

$$\xi_{n,j} = X_{\frac{j}{n}} - X_{\frac{j-1}{n}}.$$

By continuity of $t \mapsto X_t$,

$$\max_j(\xi_{n,j}) \to 0 \tag{4.1}$$

almost surely as $n \to \infty$.

Let $Y_{n,j} = \xi_{n,j} \mathbb{1}_{|\xi_{n,j}|\leq 1}$. Then $\sum_{j=1}^{n} Y_{n,j} \to X_1$ almost surely by (4.1). Let $\tilde{Y}_{n,j}$ be a symmetrization of $Y_{n,j}$, that is, take two independent copies $Y_{n,j}^{(1)}$ and $Y_{n,j}^{(2)}$ of $Y_{n,j}$, $j = 1, \ldots, n$, and set $\tilde{Y}_{n,j} = Y_{n,j}^{(1)} - Y_{n,j}^{(2)}$. Now $Y_{n,j}$ and $\tilde{Y}_{n,j}$ have finite moments and hence

$$m_n = \sum_{j=1}^{n} \mathbb{E}\, Y_{n,j}, \qquad s_n = \sum_{j=1}^{n} \operatorname{Var} Y_{n,j} \tag{4.2}$$

are finite. In addition

$$\sum_{j=1}^{n} \mathbb{E}\, \tilde{Y}_{n,j} = 0, \qquad s_n^{-1} \sum_{j=1}^{n} \mathbb{E}\left(\tilde{Y}_{n,j}^2\right) = 2. \tag{4.3}$$

Since $\sum_{j=1}^{n} \tilde{Y}_{n,j}$ converges almost surely to the symmetrization of X_1, s_n has to be bounded. Otherwise along some subsequence $s_{n_k} \nearrow \infty$ and $s_{n_k}^{-1/2} \sum_{j=1}^{n_k} \tilde{Y}_{n_k,j}$ converges weakly (in distribution) to a $N(0,2)$ distributed random variable by a version of central limit theorem (CLT). This is a contradiction. When using the CLT we need both (4.1) and (4.3), see Kallenberg [7] Theorem 5.15.

Since s_n is bounded we can choose a subsequence s_{n_k} which converges to $s \geq 0$. Then again by CLT, $\sum_{j=1}^{n_k} (Y_{n_k,j} - \mathbb{E}\, Y_{n_k,j})$ converges to a $N(0,s)$ distributed random variable. Therefore m_{n_k} converges to some m. Hence X_1 is distributed according to the normal distribution $N(m,s)$. □

Now the driving process of a random curve γ distributed according to $\mathbb{P}^{(\mathbb{H},0,\infty)}$ is

$$W_t = \sqrt{\kappa} B_t + \alpha t$$

for some $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. We will show that $\alpha = 0$. We apply once more CI and note that $\mathbb{P}^{(\mathbb{H},0,\infty)}$ is invariant under the scaling $z \mapsto \lambda z$, $\lambda > 0$, that is, $\gamma^{(\lambda)}(t) = \lambda\gamma(t/\lambda^2)$ is distributed as γ. Note that the correct parametrization of $\gamma^{(\lambda)}$ follows from the scaling property of the half-plane capacity. Now by a similar calculation as above, it follows that the driving process of $\gamma^{(\lambda)}$ is $(W_t^{(\lambda)})_{t\in\mathbb{R}_+}$ is

$$W_t^{(\lambda)} = \lambda W_{t/\lambda^2}.$$

Since $(W_t^{(\lambda)})_{t\in\mathbb{R}_+}$ is distributed as $(W_t)_{t\in\mathbb{R}_+}$, the driving process satisfies the Brownian scaling and hence $\alpha = 0$ and

$$W_t = \sqrt{\kappa} B_t.$$

To conclude this section, we have shown that the only families $(\mathbb{P}^{U,a,b})$ satisfying CI and the DMP are those where the measure $\mathbb{P}^{(\mathbb{H},0,\infty)}$ is the law of a random curve with a Loewner driving process equal to a constant multiple of a one-dimensional Brownian motion.

4.1.2 Definition of SLE as a stochastic Loewner chain

By Lemma 3.2.6, the mapping from the continuous functions $(W_t)_{t\in\mathbb{R}_+}$ to the Loewner chains $(g_t)_{t\in\mathbb{R}_+}$ is continuous in the following sense (the type of convergence giving the topology for the conformal maps is called Carathéodory convergence): if a sequence $(W_t^{(n)})_{t\in\mathbb{R}_+}$ converges to $(W_t)_{t\in\mathbb{R}_+}$ in the sense that for each $T > 0$, $\sup_{0\leq t\leq T} |W_t^{(n)} - W_t| \to 0$ as $n \to \infty$, then for each $T > 0$, and for each compact $J \subset \mathbb{H} \setminus K_T$, $g_t^{(n)}(z)$ converges to $g_t(z)$ uniformly in $z \in J$ and $t \in [0, T]$. Especially that map from driving terms to the Loewner chains is measurable and hence if we have probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a continuous stochastic process $(W_t)_{t\in\mathbb{R}_+}$, we can define a Loewner chain valued random variable $(g_t)_{t\in\mathbb{R}_+}$ correponding to the driving term $(W_t)_{t\in\mathbb{R}_+}$.

Next definition is the simplest version of the definition of SLE. We will later redefine SLE as soon as we know that SLE is a random curve

Definition 4.1.2. Let $\kappa \geq 0$. A *chordal Schramm–Loewner evolution* SLE(κ) is a random Loewner chain (the solution of (3.11)) with a driving process $(W_t)_{t\in\mathbb{R}_+}$ equal to a Brownian motion with variance parameter κ, that is, $W_t = \sqrt{\kappa} B_t$ where $(B_t)_{t\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion.

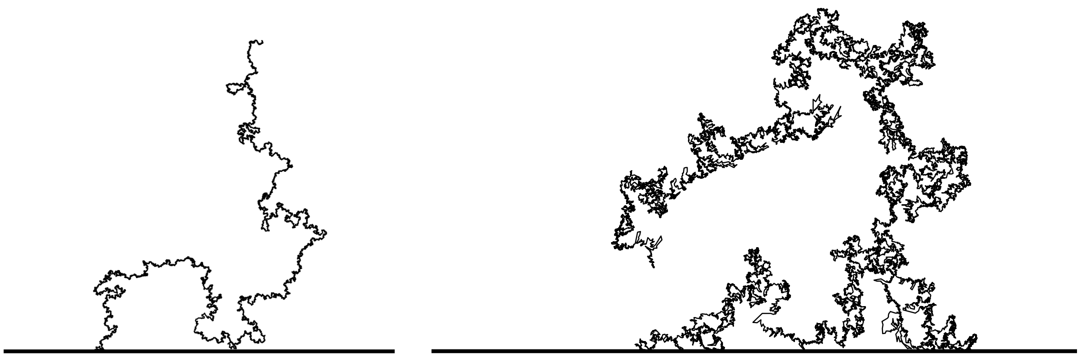

Figure 4.1: Realizations of hulls of SLE(3) and SLE(6) drawn up to a finite time. As it turns out these curves will tend to infinity as the time tends infinity.

Remark. We call this kind of SLEs *chordal* because we expect that they will be random curves that connect two boundary points, namely, 0 and ∞. A *radial* SLE would be a random curve connecting a boundary point to an interior point.

Example 4.1.3. Consider SLE(0). Then g_t is the solution of the equation

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \qquad g_0(z) = z$$

which can be integrated to give $g_t(z) = \sqrt{z^2 + 4t}$. Therefore SLE(0) is the vertical line segment $t \mapsto 2\sqrt{t}$. See also Example 3.2.5. Since this example is trivial, we assume now always that $\kappa > 0$.

Theorem 4.1.4. *Let $(K_t)_{t\in\mathbb{R}_+}$ be SLE(κ), $\kappa > 0$, and $(W_t)_{t\in\mathbb{R}_+}$ the corresponding driving process which is a Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. SLE($\kappa$) satisfies the following properties.*

- ***Scale invariance:*** *For any $\lambda > 0$, $(\lambda K_{t/\lambda^2})_{t\in\mathbb{R}_+} \stackrel{d}{=} (K_t)_{t\in\mathbb{R}_+}$.*
- ***Conformal Markov property:*** *For any $s \in \mathbb{R}_+$, the family of hulls*

$$(\hat{K}_{s,t})_{t\in\mathbb{R}_+} = \left(\overline{g_s(K_{s+t} \setminus K_s) - W_s}\right)_{t\in\mathbb{R}_+}$$

  is independent of $\mathcal{F}_s$ and $(\hat{K}_{s,t})_{t\in\mathbb{R}_+} \stackrel{d}{=} (K_t)_{t\in\mathbb{R}_+}$.
- ***Strong conformal Markov property:*** *For any almost surely finite stopping time τ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, the family of hulls*

$$(\hat{K}_{\tau,t})_{t\in\mathbb{R}_+} = \left(\overline{g_\tau(K_{\tau+t} \setminus K_\tau) - W_\tau}\right)_{t\in\mathbb{R}_+}$$

  is independent of $\mathcal{F}_\tau$ and $(\hat{K}_{\tau,t})_{t\in\mathbb{R}_+} \stackrel{d}{=} (K_t)_{t\in\mathbb{R}_+}$.

Proof. The conformal maps associated to the hulls

$$\lambda K_{t/\lambda^2}, \qquad \overline{g_s(K_{s+t} \setminus K_s) - W_s}, \qquad \overline{g_\tau(K_{\tau+t} \setminus K_\tau) - W_\tau}$$

are

$$\lambda g_{t/\lambda^2}(z/\lambda), \qquad \hat{g}_{s,t}(z), \qquad \hat{g}_{\tau,t}(z),$$

respectively, where

$$\hat{g}_{s,t}(z) = g_{s+t} \circ g_s^{-1}(z + W_s) - W_s.$$

By differentiating these functions with respect to t, we find that they satisfy the Loewner equation with the driving processes

$$\lambda W_{t/\lambda^2}, \qquad W_{s+t} - W_s, \qquad W_{\tau+t} - W_\tau,$$

respectively. The claims now follow from the scaling property, the Markov property and the strong Markov property of Brownian motion. □

Exercise. Define $m(z) = -\bar{z}$ which is an injective antiholomorphic self-map of $\mathbb{H}$. Show that SLE(κ), $\kappa > 0$, is *symmetric*, i.e., $(m(K_t))_{t\in\mathbb{R}_+} \stackrel{\mathrm{d}}{=} (K_t)_{t\in\mathbb{R}_+}$. Is the random Loewner chain with the driving process $W_t = \sqrt{\kappa} B_t + \alpha t$ symmetric?

Guided by the Schramm's principle, now that we know how to define $\mathbb{P}^{(\mathbb{H},0,\infty)}$ we would like to define $\mathbb{P}^{(U,a,b)}$. It is natural to use the conformal invariance requirement for doing this and define SLE(κ) in other domains by the conformal image of a SLE(κ) in $\mathbb{H}$. Actually this definition relies on the fact that SLE(κ) in $\mathbb{H}$ started from $W_0 = 0$ has scale invariance, otherwise the law of SLE(κ) in the other domain would depend on the choice of the conformal map.

Definition 4.1.5. Let $(K_t)_{t\in\mathbb{R}_+}$ be a (chordal) SLE(κ) and let U be a simply connected domain and a and b two boundary points of U with $a \neq b$. We define *(chordal) SLE(κ) in a domain U going from a to b* to be the image of $(K_t)_{t\in\mathbb{R}_+}$ under any conformal onto map $\phi : \mathbb{H} \to U$ with $\phi(0) = a$ and $\phi(\infty) = b$.

Remark. This definition is unique only up to a linear time change, because all the conformal onto maps from $\mathbb{H}$ to U with the above properties are of the form $z \mapsto \phi(\lambda z)$ where $\lambda > 0$ is a constant. By the scaling property of SLE, the choice of this conformal map only affects the time parametrization of the hulls in U.

Remark. If the boundary of U is not locally connected and ϕ doesn't extend continuously to the boundary, a and b has to be understood as "generalized boundary points", more specifically as prime ends.

Naturally we make here the exception that SLE(κ) in $\mathbb{H}$ from x to ∞ will always have parametrization with the half-plane capacity and therefore it is defined as the solution of the Loewner equation with the driving process $W_t = x + \sqrt{\kappa} B_t$.

4.2 Phases of SLE

In this section *we assume that SLE(κ) is a random curve.* More precisely, almost surely there is a curve γ, such that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \in \mathbb{R}_+$. We will prove this result later, see Theorem 4.3.1 below.

The next theorem summarizes the important facts about the random curve γ. We will prove those statements at least partly in this section and we are going to do it in several stages.

Theorem 4.2.1. *Let the random curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ be SLE(κ) (in the sense of Theorem 4.3.1). Then*

- *For all $0 < \kappa \le 4$, γ is simple, $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$.*
- *For all $4 < \kappa < 8$, γ is not simple:*

$$\text{for any } 0 \le t_1 < t_2 \text{ there exists } t_1 < s_1 < s_2 < t_2 \text{ such that } \gamma(s_1) = \gamma(s_2). \tag{4.4}$$

  However, γ is not space-filling: for any $z \in \mathbb{H}$, $\mathbb{P}(\operatorname{dist}(z, \gamma[0,\infty)) > 0) = \mathbb{P}(z \notin \gamma[0,\infty)) = 1$.

- *For all $\kappa \ge 8$, γ is not simple, it satisfies 4.4, but γ is space-filling: $\mathbb{P}(z \in \gamma[0,\infty)) = 1$.*

Moreover, γ is transient in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$.

Remark. Again, there is a connection from the transience of γ to Schramm's principle where we assumed that $\mathbb{P}^{(U,a,b)}$ is the law of a random curve such that $\gamma(0) = a$ and $\lim_{t\to\infty} \gamma(t) = b$.

4.2.1 Phase transition at $\kappa = 4$

Actually, many of the properties stated in the previous theorem are consequences of a simple observation. Fix $z \in \overline{\mathbb{H}}$ with $z \neq 0$ for a moment. Let g_t be SLE(κ) with a driving process $W_t = -\sqrt{\kappa} B_t$. Define

$$\hat{Z}_t = g_t(z) - W_t, \qquad Z_t = \hat{Z}_t / \sqrt{\kappa}.$$

By the Loewner equation, these processes have the Itô differentials

$$\mathrm{d}\hat{Z}_t = \frac{2}{\hat{Z}_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \mathrm{d}Z_t = \frac{2/\kappa}{Z_t}\mathrm{d}t + \mathrm{d}B_t.$$

Therefore $(Z_t)_{t\in[0,\tau(z))}$, where $\tau(z)$ is as in the section 3.2.2, could be called as a *$\delta(\kappa)$-dimensional complex Bessel process sent from $z/\sqrt{\kappa}$* where

$$\delta(\kappa) = 1 + \frac{4}{\kappa} \in (1,\infty).$$

In the next, proposition we list some properties of the (real) Bessel process.

Proposition 4.2.2. *Let $\delta \in \mathbb{R}$ and let $(X_t)_{t\in[0,T)}$ be a δ-dimensional Bessel process sent from $x > 0$, that is, $(X_t)_{t\in[0,T)}$ is the unique solution of*

$$\mathrm{d}X_t = \frac{\delta - 1}{2X_t}\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x$$

and $T \in (0, \infty]$ is the maximal time such that the solution exists and is positive. Then

- $\mathbb{P}(T < \infty) = 1$ *if and only if* $\delta < 2$,
- $\mathbb{P}(T = \infty) = 1$ *if and only if* $\delta \geq 2$,
- $\mathbb{P}(\inf_{0\leq t<T} X_t > 0) = 1$ *if and only if* $\delta > 2$.

Remark. As we saw earlier, the Euclidian norm of a d-dimensional Brownian motion (sent away from the origin) is a d-dimensional Bessel process. In the case $\delta = 2$, the Bessel process will get arbitrarily large and small values, but it won't hit zero.

Proof. These claim can be proven for instance based on the fact that $X_t^{2-\delta}$, $\delta \neq 2$, is a local martingale and for $\delta = 2$, $\log X_t$ is a martingale. We leave the details as an exercise. □

Since $x \in \bigcup_{t\in\mathbb{R}_+} K_t$ if and only if $\tau(x) < \infty$ where $\tau(x)$ is as in the section 3.2.2, if we apply the previous result for $(g_t(x) - W_t)/\sqrt{\kappa}$, $x \in \mathbb{R} \setminus \{0\}$, we see the following properties hold.

Proposition 4.2.3. *For $0 < \kappa \leq 4$, $(\bigcup_{t\in\mathbb{R}_+} K_t) \cap \mathbb{R} = \{0\}$ and for $\kappa > 4$, $\mathbb{R} \subset \bigcup_{t\in\mathbb{R}_+} K_t$.*

Let's first show that SLE(κ), $0 < \kappa \leq 4$, is simple, based on this result. Let $s > 0$ and let x_- and x_+ be the two images of 0 under the map $g_s - W_s$. By the previous proposition and by the conformal Markov property, $\hat{\gamma}(t) = g_s(\gamma(s + t)) - W_s$, $t \in \mathbb{R}_+$, intersect the real axis only at 0. Especially it doesn't intersect $[x_-, 0) \cup (0, x_+]$. Since $f_s = g_s^{-1}$ is continuous to the boundary, this implies that

$$\gamma[0, s] \cap \gamma[s, \infty) = \{\gamma(s)\} \tag{4.5}$$

almost surely. In fact this holds almost surely for all s (we can show it first for all rational s and then by continuity to all s). If $t_1 < t_2$ are such that $\gamma(t_1) = \gamma(t_2)$, then pick $t_1 < s < t_2$ such that $\gamma(s) \neq \gamma(t_1)$ which contradicts (4.5). Hence γ is simple.

Let's then show that SLE(κ), $\kappa > 4$, is not simple. Let $0 \leq s_1 < u < s_2$. Let $\hat{x}_- \leq 0 \leq \hat{x}_+$ be such that the image of $\gamma[s_1, u]$ under $g_u - W_u$ is $[\hat{x}_-, \hat{x}_+]$. Since for fixed $t > 0$, $\mathbb{P}(\tau(x) \leq t) \to 1$ as $x \searrow 0$,

$$\mathbb{P}(\gamma[0, t] \cap (0, x] \neq \emptyset) = 1$$

for all $t > 0$ and $x > 0$. Hence we can find $u < t_2 < s_2$ such that $g_u(\gamma(t_2)) - W_u \in [\hat{x}_-, 0) \cup (0, \hat{x}_+]$. And hence there exists $s_1 \leq t_1 < u$ such that $\gamma(t_1) = \gamma(t_2)$ and we have shown the property (4.4).

Now we have shown the claims in Theorem 4.2.1 about the "phase transition" at $\kappa = 4$. We will verify the claims about whether or not γ is the space-filling, in later sections. We conclude this section by proving the transience of γ in the case $0 < \kappa \leq 4$.

Let $0 < \kappa \leq 4$. Since γ doesn't hit interval $[1, 2]$, say, it is clear that $\mathrm{dist}(\gamma[0, t], [1, 2]) > 0$ for any $t > 0$, but it might be possible that $\gamma(t_k) \to [1, 2]$ along some sequence $t_k \to \infty$. The next result shows that this doesn't happen.

Proposition 4.2.4. *When $0 < \kappa \leq 4$, $\mathbb{P}(\mathrm{dist}(\gamma[0, \infty), [x, x']) > 0) = 1$ for any $0 < x < x'$ or $x < x' < 0$.*

Proof. By symmetry and the scale invariance of SLE(κ), it is enough to show that $\mathbb{P}(\mathrm{dist}(\gamma[0, \infty), [1, x]) > 0) = 1$ for all $x > 1$. Let $0 < \delta < 1/4$ and define

$$\sigma_\delta = \inf\{t \in \mathbb{R}_+ \,:\, \mathrm{dist}(\gamma(t), [1, x]) \leq \delta\}. \tag{4.6}$$

Let's consider the event $\sigma_\delta < \infty$. Let $R > 2x$ be such that $\gamma[0, \sigma_\delta] \subset B(0, R)$. Let $h(z)$ be the bounded harmonic function on $\mathbb{H} \setminus \gamma[0, \sigma_\delta]$ that gets value 1 on the right-hand side of $\gamma[0, \sigma_\delta]$ and on $[0, 1/2]$

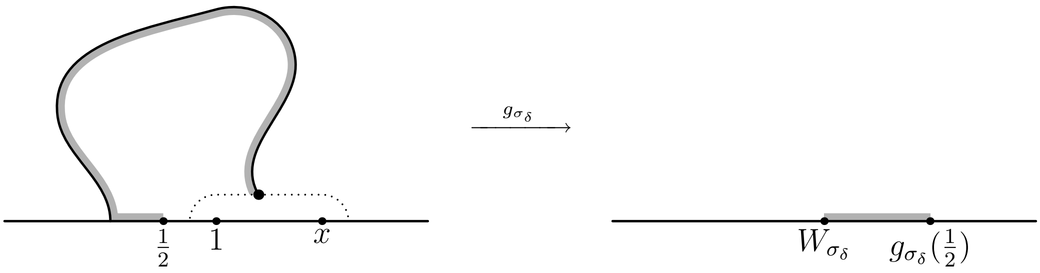

Figure 4.2: The harmonic function h in the proof of Proposition 4.2.4 has boundary value 1 in the shaded boundary arc and 0 elsewhere on the boundary. The dotted curve is the set of points at distance δ from the interval $[1, x]$.

and the boundary values are 0 elsewhere. See Figure 4.2. Then by applying the conformal map g_{σ_δ} we see that

$$h(iy) = \frac{1}{\pi} \int_{W_{\sigma_\delta}}^{g_{\sigma_\delta}(1/2)} \frac{\operatorname{Im} g_{\sigma_\delta}(iy)}{|g_{\sigma_\delta}(iy) - \xi|^2} \mathrm{d}\xi = \frac{1}{\pi y}(g_{\sigma_\delta}(1/2) - W_{\sigma_\delta}) + o(1)$$

as $y \to \infty$. On the other hand, we can formulate $h(iy)$ as the probability that a complex Brownian motion sent from iy exits $\mathbb{H} \setminus \gamma[0, \sigma_\delta]$ through the right-hand side of $\gamma[0, \sigma_\delta]$ or the interval $[0, 1/2]$. On this event the Brownian motion has to intersect the vertical line segment from the interval $[1, x]$ to $\gamma(\sigma_\delta)$. Let $x_0 = \operatorname{Re} \gamma(\sigma_\delta)$. Then the probability that a complex Brownian motion sent from iy will hit the segment $[x_0, x_0 + i\delta]$ before exiting the upper half-plane is equal to $\frac{2\delta}{\pi y} + o(1)$ as $y \to \infty$ again by considering the conformal associated to the vertical line segment together with the Poisson kernel of the upper half-plane.

Therefore there is a constant $C > 0$ such that on the event $\sigma_\delta < \infty$ $g_{\sigma_\delta}(1/2) - W_{\sigma_\delta} \leq C\delta$. Because the infimum of a Bessel process is positive, there exists a random δ_0 such that $\sigma_\delta = \infty$ for $0 < \delta < \delta_0$. □

Proposition 4.2.5. *For $0 < \kappa \leq 4$, $|\gamma(t)| \to \infty$ as $t \to \infty$.*

Proof. First of all $|\gamma(t_k)| \to \infty$ along some sequence $t_k \to \infty$, because otherwise γ would be bounded and hence had bounded half-plane capacity.

Let T_1 be the hitting time of $\partial B(0, 1)$ by γ and let $x_- < 0 < x+$ be the two images of 0 under the map $g_{T_1} - W_{T_1}$. Then by Proposition 4.2.4 almost surely distance from the SLE(κ) curve $\hat{\gamma}(t) = g_{T_1}(\gamma(T_1 + t)) - W_{T_1}$, $t \in \mathbb{R}_+$, to $[-n, -1/n] \cup [1/n, n]$ is positive for all $n \in \mathbb{N}$. Hence it will stay at a positive distance from x_- and x_+ and consequently, there exists a random variable $r > 0$ such that $|\gamma(t)| \geq r$ for $t \geq T_1$. Using this property and scaling we can construct a sequence of random variables $0 < R_1 < R_2 < \ldots$ such that $R_k \to \infty$ almost surely and γ doesn't enter to $B(0, R_{k-1})$ after hitting $\partial B(0, R_k)$. □

4.2.2 Phase transition at $\kappa = 8$

The property whether or not $x \in \bigcup_{t \in \mathbb{R}_+} K_t$, for $x \in \mathbb{R} \setminus \{0\}$, relied completely on the properties of a single real Bessel process. This section provides an example of a property where we have to consider $g_t(z)$ for two different points z simultaneously. In this section, we assume that $\kappa > 4$.

Let $0 < \xi_0^{(1)} < \xi_0^{(2)}$ and $\xi_t^{(k)} = g_t(\xi_0^{(k)}) - W_t$, $k = 1, 2$, where $W_t = -\sqrt{\kappa}\mathrm{d}B_t$. Now

$$\mathrm{d}\xi_t^{(k)} = \frac{2}{\xi_t^{(k)}}\, \mathrm{d}t + \sqrt{\kappa}\, \mathrm{d}B_t.$$

Denote the law of the pair $(\xi_t^{(1)}, \xi_t^{(2)})$ by $\mathbb{P}^{\xi_0^{(1)}, \xi_0^{(2)}}$. Let $T^{(k)}$ be the hitting time of 0 by $\xi_t^{(k)}$. Then $T^{(1)} \leq T^{(2)} < \infty$ by the results of the previous section. We would like to resolve, whether or not $T^{(1)} < T^{(2)}$, and to calculate the function

$$\hat{F}\left(\xi_0^{(1)}, \xi_0^{(2)}\right) = \mathbb{P}^{\xi_0^{(1)}, \xi_0^{(2)}}\left(T^{(1)} < T^{(2)}\right)$$

By scale invariance we can write

$$\hat{F}\left(\xi_0^{(1)}, \xi_0^{(2)}\right) = F\left(\frac{\xi_0^{(1)}}{\xi_0^{(2)}}\right).$$

By Markov property of SLE(κ), on the event $t < T^{(1)}$

$$\mathbb{P}^{\xi_0^{(1)},\xi_0^{(2)}}\left(T^{(1)} < T^{(2)} \middle| \mathcal{F}_t\right) = F\left(\frac{\xi_t^{(1)}}{\xi_t^{(2)}}\right).$$

Therefore

$$M_t = F\left(\frac{\xi_{t\wedge T^{(1)}}^{(1)}}{\xi_{t\wedge T^{(1)}}^{(2)}}\right)$$

is a martingale. We will investigate the consequences of this observation assuming also that F is smooth.

Define

$$X_t = \log\frac{\xi_t^{(2)}}{\xi_t^{(1)}}, \qquad S_t = \exp(-X_t) = \frac{\xi_t^{(1)}}{\xi_t^{(2)}}.$$

Then $X_t \in (0,\infty)$ and $S_t \in (0,1)$ for $t \in [0, T^{(1)})$. By Itô's formula

$$\mathrm{d}X_t = \frac{1}{2}(4-\kappa)(e^{-2X_t}-1)\frac{\mathrm{d}t}{(\xi_t^{(1)})^2} + \sqrt{\kappa}(e^{-X_t}-1)\frac{\mathrm{d}B_t}{\xi_t^{(1)}}$$

and

$$\begin{aligned}\mathrm{d}S_t &= -S_t\mathrm{d}X_t + \frac{1}{2}S_t\mathrm{d}\langle X\rangle_t \\ &= \frac{1}{2}S_t(1-S_t)\big[(4-\kappa)(S_t+1)+\kappa(1-S_t)\big]\frac{\mathrm{d}t}{(\xi_t^{(1)})^2} + \sqrt{\kappa}S_t(1-S_t)\frac{\mathrm{d}B_t}{\xi_t^{(1)}}.\end{aligned} \tag{4.7}$$

Also by Itô's formula

$$\begin{aligned}\mathrm{d}F(S_t) =& \frac{1}{2}S_t(1-S_t)\left\{\big[(4-\kappa)(S_t+1)+\kappa(1-S_t)\big]F'(S_t) + \kappa S_t(1-S_t)F''(S_t)\right\}\frac{\mathrm{d}t}{(\xi_t^{(1)})^2} \\ &+ \{\ \dots\ \}\mathrm{d}B_t\end{aligned}$$

If $F(S_t)$ is a local martingale then

$$(\log F')'(s) = \frac{F''(s)}{F'(s)} = \frac{(\kappa-4)(s+1)-\kappa(1-s)}{\kappa s(1-s)} = \frac{2(\kappa-4)}{\kappa(1-s)} - \frac{4}{\kappa s}$$

and hence

$$F(s) = C\int(1-s)^{-\frac{2(\kappa-4)}{\kappa}}s^{-\frac{4}{\kappa}}\mathrm{d}s + C'$$

The integral is convergent at 0 if and only if $\kappa > 4$ and at 1 if and only if $\kappa < 8$.

For any $4 < \kappa < 8$, define a function

$$F(s) = C(\kappa)\int_s^1 (1-u)^{-\frac{2(\kappa-4)}{\kappa}}u^{-\frac{4}{\kappa}}\mathrm{d}u \tag{4.8}$$

where the constant is such that $F(0) = 1$, that is,

$$C(\kappa) = \left(\int_0^1 (1-u)^{-\frac{2(\kappa-4)}{\kappa}}u^{-\frac{4}{\kappa}}\mathrm{d}u\right)^{-1} = \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\Gamma(8/\kappa-1)}.$$

This F is now the function that we were searching for as we'll see soon.

It is possible to show that S_t will exit any compact subinterval of $(0,1)$ before $T^{(1)}$ by making a time-change $\mathrm{d}s = (\xi_t^{(1)})^{-2}\mathrm{d}t$ in (4.7) and then comparing to a Brownian motion. We skip the details of this argument. Define

$$G(s) = \int_0^s (1-u)^{-\frac{2(\kappa-4)}{\kappa}} u^{-\frac{4}{\kappa}} \mathrm{d}u$$

then $G : [0,1) \to [0,\infty)$ is strictly increasing and $\lim_{s\nearrow 1} G(s) < \infty$ if and only if $4 < \kappa < 8$. Let $0 < s < s' < 1$. The optional stopping theorem shows that

$$G(s) = G(s') + (G(\varepsilon) - G(s'))\mathbb{P}^{s,1}\left((S_t)_{t\in[0,T^{(1)})} \text{ hits } \varepsilon \text{ before } s'\right)$$
$$\implies \quad \mathbb{P}^{s,1}\left((S_t)_{t\in[0,T^{(1)})} \text{ hits all } \varepsilon\in(0,s) \text{ before } s'\right) = 1 - \frac{G(s)}{G(s')}. \tag{4.9}$$

Hence $\mathbb{P}^{s,1}(\inf_{t\in[0,T^{(1)})} S_t = 0) = 1$ when $\kappa \geq 8$ and $0 < \mathbb{P}^{s,1}(\inf_{t\in[0,T^{(1)})} S_t = 0) < 1$ when $4 < \kappa < 8$.

Lemma 4.2.6. *The event that* $\inf_{t\in[0,T^{(1)})} S_t = 0$ *and the event that* $T^{(1)} < T^{(2)}$ *differ only with a event of zero probability.*

Proof. If $T^{(1)} < T^{(2)}$ then clearly $\lim_{t\to T^{(1)}} S_t = 0$.

Let $\sigma_\varepsilon = \inf\{t \in [0,T^{(1)}) \,:\, S_t = \varepsilon\}$. If $\inf_{t\in[0,T^{(1)})} S_t = 0$, then $\sigma_\varepsilon < T^{(1)}$ for all $0 < \varepsilon < s$. Now use strong Markov property of $(\xi_t^{(1)}, \xi_t^{(2)})$ and scale invariance to see that

$$\mathbb{P}^{(s,1)}\left(\left\{T^{(1)} = T^{(2)}\right\} \cap \left\{\tau_\varepsilon < T^{(1)}\right\}\right) = \mathbb{E}^{(s,1)}\left(\mathbb{P}^{(s,1)}\left(T^{(1)} = T^{(2)} \,\middle|\, \mathcal{F}_{\sigma_\varepsilon}\right) \,;\, \left\{\tau_\varepsilon < T^{(1)}\right\}\right)$$
$$\leq \mathbb{P}^{(\varepsilon^{1/2},\varepsilon^{-1/2})}\left(T^{(1)} = T^{(2)}\right) \to 0$$

as $\varepsilon \searrow 0$, because

$$\mathbb{P}^{(\varepsilon^{1/2},\varepsilon^{-1/2})}\left(T^{(1)} < t < T^{(2)}\right) \to 1$$

for any fixed $t > 0$. Hence

$$\mathbb{P}^s\left(\left\{T^{(1)} = T^{(2)}\right\} \cap \left\{\inf_{t\in[0,T^{(1)})} S_t = 0\right\}\right) = 0$$

which gives the claim. □

We have now shown the following result. For the claim for $4 < \kappa < 8$, we use (4.9).

Proposition 4.2.7. *For* $\kappa \geq 8$, $\mathbb{P}(\tau(x) < \tau(x')) = 1$ *for all* $0 < x < x'$. *For* $4 < \kappa < 8$, $\mathbb{P}(\tau(x) < \tau(x')) = F(x/x')$ *for all* $0 < x < x'$ *where* F *is as in* (4.8).

4.2.3 Time-change of semimartingales

The next result about time-change of stochastic integrals is an addition to the tools introduced in Chapter 1. It has several application to SLE.

Proposition 4.2.8. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *and let* $(B_t)_{t\in\mathbb{R}_+}$ *be a standard one-dimensional Brownian motion with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. *Let* $a(t,\omega)$ *be a continuous, positive, adepted process. Define a random time-change by setting:*

$$S(t,\omega) = \int_0^t a(r,\omega)^2 \,\mathrm{d}r, \qquad \sigma(s,\omega) = \inf\{t \in \mathbb{R}_+ \,:\, S(t,\omega) \geq s\}$$

Let $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ *be the process defined by*

$$\tilde{B}_s(\omega) = \int_0^{\sigma(s)} a(r,\omega) \,\mathrm{d}B_r(\omega),$$

which is a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$*. Then for any continuous, adapted process* $v(t,\omega)$ *the following time-change formula holds*

$$\int_0^s v(\sigma(q),\omega)\,\mathrm{d}\tilde{B}_q(\omega) = \int_0^{\sigma(s)} v(r,\omega)a(r,\omega)\,\mathrm{d}B_r(\omega).$$

Moreover if X_t *is a semimartingale*

$$\mathrm{d}X_t(\omega) = u(t,\omega)\mathrm{d}t + v(t,\omega)\mathrm{d}B_t(\omega)$$

then the process $(\tilde{X}_s)_{s\in\mathbb{R}_+}$ *defined by* $\tilde{X}_s = X_{\sigma(s)}$ *is a semimartingale with respect to* $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$ *and* $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ *and satisfies*

$$\mathrm{d}\tilde{X}_s = \frac{u(\sigma(s))}{a(\sigma(s))^2}\mathrm{d}s + \frac{v(\sigma(s))}{a(\sigma(s))}\mathrm{d}\tilde{B}_s.$$

Proof. Exercise. □

4.2.4 SLE(κ), $\kappa \geq 8$, is space filling

We will also state a proposition here about whether or not $z \in \bigcup_{t\in\mathbb{R}_+} K_t$.

Proposition 4.2.9. *When* $\kappa > 4$, $z \in \bigcup_{t\in\mathbb{R}_+} K_t$ *almost surely for any* $z \in \mathbb{H}$*.*

In the proof of this result we need to keep track of a single point $Z_t = g_t(z) - W_t$, $z \in \mathbb{H}$ and the proof relies on the fact that $Z_t^{1-4/\kappa}$ is a local martingale (the real and imaginary parts are local martingales). We will skip the details of the proof and leave them to the exercises. Instead, in this section we will show that $z \in \mathbb{H}$ is not swallowed when $\kappa \geq 8$. We say that z is *swallowed*, if $z \in \bigcup_{t\in\mathbb{R}_+} K_t$ but $z \notin \gamma[0,\infty)$. This result will be the first one where we consider the Loewner flow of SLE(κ) for an interior point. Therefore let's first list some formulas related to the flow.

For any fixed $z \in \overline{\mathbb{H}}$, let $Z_t = g_t(z) - W_t$, $t \in [0, \tau(z))$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively, and as usual let $W_t = -\sqrt{\kappa}B_t$. Then from the Loewner equation it follows that

$$\mathrm{d}X_t = \frac{2X_t}{X_t^2+Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t \tag{4.10}$$

$$\partial_t Y_t = -\frac{2Y_t}{X_t^2+Y_t^2} \tag{4.11}$$

$$\partial_t \log|g_t'(z)| = -2\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}. \tag{4.12}$$

The last equation follows by taking derivative of the Loewner equation with respect to z.

Since $z \mapsto \log z$ is holomorphic, using Itô's formula for the real and imaginary parts of $\log Z_t$ gives

$$\mathrm{d}\log Z_t = (2-\kappa/2)\frac{\mathrm{d}t}{Z_t^2} + \sqrt{\kappa}\frac{\mathrm{d}B_t}{Z_t}$$

and therefore by taking real and imaginary parts we find that

$$\mathrm{d}\log|Z_t| = (2-\kappa/2)\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa}\frac{X_t}{X_t^2+Y_t^2}\mathrm{d}B_t \tag{4.13}$$

$$\mathrm{d}\arg Z_t = -(2-\kappa/2)\frac{2X_tY_t}{(X_t^2+Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa}\frac{Y_t}{X_t^2+Y_t^2}\mathrm{d}B_t. \tag{4.14}$$

Let now $\theta_t = \arg Z_t$. Then we can rewrite the previous equation as

$$\mathrm{d}\theta_t = \frac{1}{2}(\kappa-4)\sin(2\theta_t)\frac{\mathrm{d}t}{X_t^2+Y_t^2} - \sqrt{\kappa}\sin(\theta_t)\frac{\mathrm{d}B_t}{\sqrt{X_t^2+Y_t^2}}$$

Now observe the following fact which shows that we should study θ_t in more detail.

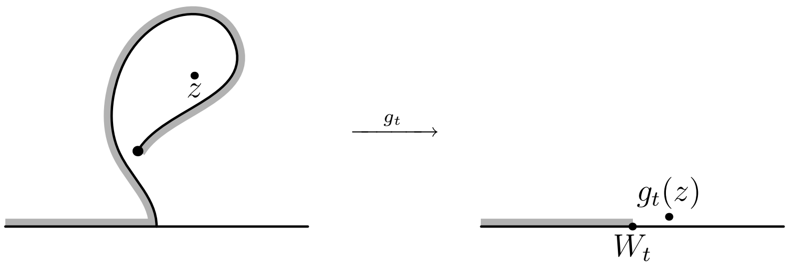

Figure 4.3: The harmonic measure of the shaded boundary arc on the left picture can be formulated by the conformal map g_t as a constant multiple of the argument of $g_t(z) - W_t$. Therefore as z is swallowed $\arg(g_t(z) - W_t)$ converges to 0 or π.

Lemma 4.2.10. *If $z \in \mathbb{H}$ is swallowed, that is, $\tau(z) < \infty$ but $z \notin \gamma[0, \infty)$, then $\lim_{t \to \tau(z)} \theta_t$ exists and equals to 0 or π.*

Proof. This is a simple application of the harmonic measure. Define $h_t(z)$ as the bounded harmonic function in $H_t = \mathbb{H} \setminus K_t$ such that the boundary values are π on the union of $(-\infty, 0]$ and the left-hand side of $\gamma[0, t]$ and 0 otherwise. Then if z is swallowed, $h_t(z) \to \{0, \pi\}$ as $t \to \tau(z)$ by the Beurling estimate. On the other hand, $h_t(z) = \arg(g_t(z) - W_t)$ and the claim follows. See also Figure 4.3. □

Make a time-change

$$S(t) = \int_0^t \frac{\mathrm{d}r}{X_r^2 + Y_r^2}, \qquad \sigma(s) = S^{-1}(s), \qquad \hat{B}_s = -\int_0^{\sigma(s)} \frac{\mathrm{d}B_r}{\sqrt{X_r^2 + Y_r^2}}$$

which is implies that

$$\dot{\sigma}(s) = X_{\sigma(s)}^2 + Y_{\sigma(s)}^2.$$

Under this time-change $\hat{\theta}_s = \theta_{\sigma(s)}$ satisfies

$$\mathrm{d}\hat{\theta}_s = \frac{1}{2}(\kappa - 4)\sin(2\hat{\theta}_s)\mathrm{d}s + \sqrt{\kappa}\sin(\hat{\theta}_s)\mathrm{d}\hat{B}_s.$$

Now $\hat{Y}_s = Y_{\sigma(s)}$ safisfies

$$\partial_s \hat{Y}_s = -2\hat{Y}_s \quad \Longrightarrow \quad \hat{Y}_s = (\operatorname{Im} z)\exp(-2s).$$

Therefore $\lim_{s \to \infty} \sigma(s) = \tau(z)$ and $0 < \hat{\theta}_s < \pi$ for all $s \in \mathbb{R}_+$. Hence we now need to investigate whether or not $\lim_{s \to \infty} \hat{\theta}_s \in \{0, \pi\}$.

Similarly as above, it is possible to find a function $F : (0, \pi) \to \mathbb{R}$ such that $F(\hat{\theta}_s)$ is a local martingale. Namely, we can find smooth F with $F(\pi/2) = 0$ and $F'(\pi/2) = 1$ such that for all $\theta \in (0, \pi)$

$$\frac{F''(\theta)}{F'(\theta)} = -\frac{\kappa - 4}{\kappa}\frac{\sin(2\theta)}{\sin(\theta)^2}.$$

Since

$$\frac{\sin(2\theta)}{\sin(\theta)^2} \approx \begin{cases} \frac{2}{\theta} & \text{when } \theta \approx 0 \\ -\frac{2}{\pi - \theta} & \text{when } \theta \approx \pi \end{cases},$$

we have that $-\lim_{\theta \searrow 0} F(\theta)$ and $\lim_{\theta \nearrow \pi} F(\theta)$ are infinite if and only if $\kappa \geq 8$.

We will show that for any $\kappa \geq 8$, it is not true that $\lim_{s \to \infty} \hat{\theta}_s \in \{0, \pi\}$. By symmetry, it is enough to show that on the event that $\liminf_{s \in \mathbb{R}_+} \hat{\theta}_s = 0$ we have that $\limsup_{s \in \mathbb{R}_+} \hat{\theta}_s > 0$. Let $\sigma_\varepsilon = \inf\{s \in \mathbb{R}_+ : \hat{\theta}_s = \varepsilon\}$ for any $\varepsilon \in (0, \pi)$. Let $0 < \varepsilon' < \varepsilon < \theta$. On the event $\sigma_\varepsilon < \infty$ by strong Markov property of $\hat{\theta}_s$

$$\mathbb{P}^\theta\left((\hat{\theta}_s)_{s \geq \sigma_\varepsilon} \text{ hits } \pi/2 \text{ before } \varepsilon' \middle| \mathcal{F}_{\sigma_\varepsilon} \right) = \mathbb{P}^\varepsilon\left((\hat{\theta}_s)_{s \geq \mathbb{R}_+} \text{ hits } \pi/2 \text{ before } \varepsilon' \right).$$

The right-hand side can be given in terms of F, by the optional stopping theorem

$$F(\varepsilon) = F(\varepsilon') + (F(\pi/2) - F(\varepsilon'))\mathbb{P}^\varepsilon\left((\hat{\theta}_s)_{s\geq\mathbb{R}_+} \text{ hits } \pi/2 \text{ before } \varepsilon'\right)$$
$$\implies \qquad \mathbb{P}^\varepsilon\left((\hat{\theta}_s)_{s\geq\mathbb{R}_+} \text{ hits } \pi/2 \text{ before } \varepsilon'\right) = \frac{F(\varepsilon) - F(\varepsilon')}{F(\pi/2) - F(\varepsilon')}.$$

Since the right-hand side of the last equation goes to one as $\varepsilon' \searrow 0$, we can choose a decreasing sequence ε_n such that on the event $\sigma_{\varepsilon_n} < \infty$

$$\mathbb{P}^\theta\left((\hat{\theta}_s)_{s\geq\sigma_{\varepsilon_n}} \text{ hits } \pi/2 \text{ before } \varepsilon_{n+1} \middle| \mathcal{F}_{\sigma_{\varepsilon_n}}\right) \geq 1 - 2^{-n}$$

Therefore on the event that $\liminf_{s\in\mathbb{R}_+} \hat{\theta}_s = 0$, $\limsup_{s\in\mathbb{R}_+} \hat{\theta}_s \geq \pi/2$. Now we have shown the following result.

Proposition 4.2.11. *For any $\kappa \geq 8$, almost surely z is not swallowed.*

4.3 SLE is a random curve

Throughout this section $f_t = g_t^{-1}$. In this section we will finally prove that any chordal SLE is a random curve.

Theorem 4.3.1. *Almost surely SLE(κ) is a curve in the sense that $\gamma(t) = \lim_{\varepsilon\searrow 0} f_t(W(t) + i\varepsilon)$ exists for all $t \in \mathbb{R}_+$, $t \mapsto \gamma(t)$ is continuous and $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \in \mathbb{R}_+$.*

Strictly speaking, we only prove this for $\kappa \neq 8$. The case $\kappa = 8$ follows from a result by Lawler, Schramm and Werner by which a certain discrete random curve converges to SLE(8) as lattice mesh tends to zero.

We will first present the auxiliary results needed for the proof of Theorem 4.3.1. The function $\gamma(t) = \lim_{\varepsilon\searrow 0} f_t(W(t) + i\varepsilon)$ is called the *trace* of the Loewner chain. It is useful to define

$$\tilde{f}_t(z) = f_t(z + W_t).$$

The goal of the sections 4.3.1 and 4.3.2 is to have good bounds for $|\tilde{f}_t'(iy)|$, $t \in [0,1], y \in (0,1]$. The proof of the theorem will be given in the section 4.3.3 below.

4.3.1 Reverse Schramm–Loewner evolution

Lemma 4.3.2. *Let $h_t(z)$ be the solution of*

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z \tag{4.15}$$

where $W_t = \sqrt{\kappa}B_t$. Then the solution is well-defined for $t \in \mathbb{R}_+$ and the functions $z \mapsto f_t(z+W_t) - W_t$ and $z \mapsto h_t(z)$ have the same distribution for each $t \in \mathbb{R}_+$. Especially, $\tilde{f}_t'(z)$ has the same distribution as $h_t'(z)$.

Remark. This holds only for a single time instant. It is not true that the joint law of $f_t(z+W_t) - W_t$, $t \in \mathbb{R}_+$, is the same as the joint law of $h_t(z)$, $t \in \mathbb{R}_+$.

Proof. The solution of (4.15) is well-defined for all t because the imaginary part of $h_t(z)$ is strictly increasing for all t.

Let $W_t = \sqrt{\kappa}B_t$. Fix $s > 0$. Let $V_t = W_{s-t} - W_s$, $t \in [0,s]$. Then $(V_t)_{t\in[0,s]}$ has the same distribution as $(W_t)_{t\in[0,s]}$. Let $h_t(z)$, $t \in [0,s]$, be the solution of the differential equation

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - V_t}, \qquad h_0(z) = z.$$

It is clearly enough to show that $f_s(z+W_s)-W_s = h_s(z)$. Note that for this equality to hold $(h_t)_{t\in[0,s]}$ has to be the solution for the Brownian motion $(V_t)_{t\in[0,s]}$ not for $(W_t)_{t\in[0,s]}$.

Fix $z \in \mathbb{H}$ for a moment and let

$$\zeta_t = h_{s-t}(z - W_s) + W_s$$

for $t \in [0, s]$. Then $\zeta_0 = h_s(z - W_s) + W_s$ and

$$\dot{\zeta}_t = \frac{2}{\zeta_t - W_t}.$$

Hence $\zeta_t = g_t(\zeta_0)$ for all $t \in [0, s]$. Expecially, $z = \zeta_s = g_s(h_s(z - W_s) + W_s)$ and therefore $f_s(z + W_s) - W_s = h_s(z)$ for all $z \in \mathbb{H}$. □

4.3.2 Moments $\mathbb{E}|\tilde{f}_t'(z)|^\lambda$

Let's deal with both the forward and reverse Schramm–Loewner evolution by fixing $\nu = \pm 1$ and letting $h_t(z)$ be the solution of the following equation

$$\partial_t h_t(z) = \nu \frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z$$

where $W_t = -\sqrt{\kappa} B_t$. For fixed $z_0 = x_0 + iy_0 \in \mathbb{H}$, let $Z_t = h_t(z_0) - W_t$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively. Let's list some useful formulas which are verified in the exercises:

$$\begin{aligned}
\mathrm{d}X_t = 2\nu \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \partial_t Y_t = -2\nu \frac{Y_t}{X_t^2 + Y_t^2},\\
\partial_t |h_t'(z_0)| = -2\nu |h_t'(z_0)| \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}, \qquad \partial_t \frac{|h_t'(z_0)|}{Y_t} = 4\nu \frac{|h_t'(z_0)|}{Y_t} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2},\\
\mathrm{d} \arg Z_t = (\kappa - 4\nu) \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{Y_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,\\
\mathrm{d} \log |Z_t| = -\frac{1}{2}(\kappa - 4\nu) \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,\\
\mathrm{d} \sin \arg Z_t = (\sin \arg Z_t) \left[\frac{(\kappa - 4\nu)X_t^2 - \frac{\kappa}{2}Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t \right].
\end{aligned}$$

Now we fix $\nu = -1$. Then all the processes above are well-defined for all $t \in \mathbb{R}_+$.

Let $p, q, r \in \mathbb{R}$ and define

$$M_t = |h_t'(z_0)|^p \, Y_t^q \, (\sin \arg Z_t)^{-2r}.$$

By Itô's formula, M_t is a local martingale if and only if

$$q = p - \frac{\kappa}{2}r, \qquad r^2 - \left(1 + \frac{4}{\kappa}\right) r + \frac{2}{\kappa}p = 0$$

and in that case

$$\mathrm{d}M_t = 2r\sqrt{\kappa} \, \frac{X_t}{X_t^2 + Y_t^2} \, M_t \, \mathrm{d}B_t.$$

The reader can verify these claims.

Next we define a time-change:

$$S(t) = \int_0^t \frac{\mathrm{d}u}{X_u^2 + Y_u^2}, \qquad \sigma(s) = S^{-1}(s).$$

Let $\hat{\mathcal{F}}_s = \mathcal{F}_{\sigma(s)}$. Then

$$\hat{B}_s = \int_0^{\sigma(s)} \frac{\mathrm{d}B_u}{\sqrt{X_u^2 + Y_u^2}}$$

is a standard one-dimensional Brownian motion with respect to the filtration $(\hat{\mathcal{F}}_s)_{s\in\mathbb{R}_+}$. Denote the time-changed processes by

$$\hat{Z}_s = Z_{\sigma(s)}, \qquad \hat{X}_s = X_{\sigma(s)}, \qquad \hat{Y}_s = Y_{\sigma(s)}, \qquad \hat{h}_s(z_0) = h_{\sigma(s)}(z_0).$$

Notice that the equations

$$\partial_s \hat{Y}_s = 2\hat{Y}_s, \qquad \partial_s \frac{|\hat{h}'_s(z_0)|}{\hat{Y}_s} = -4\frac{|\hat{h}'_s(z_0)|}{\hat{Y}_s}(\sin\arg\hat{Z}_s)^2$$

hold and therefore

$$\hat{Y}_s = y_0 e^{2s} \tag{4.16}$$

$$|\hat{h}'_s(z_0)| = \exp\left(2s - 4\int_0^s (\sin\arg\hat{Z}_u)^2 \mathrm{d}u\right). \tag{4.17}$$

By (4.16), $\hat{Y}_s$ is deterministic and strictly increasing. The equation (4.17) implies that

$$e^{-2s'} \leq \frac{|\hat{h}'_{s+s'}(z_0)|}{|\hat{h}'_s(z_0)|} \leq e^{2s'}. \tag{4.18}$$

Observe also that

$$Y_t \leq \sqrt{y_0^2 + 4t} \tag{4.19}$$

This shows that $y_0 e^{2s} \leq \sqrt{y_0^2 + 4\sigma(s)}$ and hence $\sigma(s) \to \infty$ as $s \to \infty$.

Under this time-change, the local martingale $\hat{M}_s = M_{\sigma(s)}$ satisfies

$$\mathrm{d}\hat{M}_s = 2r\sqrt{\kappa}\,(\cos\arg\hat{Z}_s)\,\hat{M}_s\,\mathrm{d}B_s.$$

It is not hard to show that $(\hat{M}_s)_{s\in\mathbb{R}_+}$ is a martingale.

Lemma 4.3.3. *Let N_0 be a constant and let $(N_t)_{t\in\mathbb{R}_+}$ be a local martingale with*

$$N_t = N_0 + \int_0^t A_s N_s \mathrm{d}B_s.$$

If for every $t > 0$ there is a constant $c(t)$ such that $|A_s| \leq c(t)$ for all $s \in [0,t]$, then N_t is a martingale.

Proof. Let $M_t = N_t - N_0$. Then

$$M_t = \int_0^t (A_s M_s + A_s N_0)\mathrm{d}B_s.$$

Let $n \in \mathbb{N}$ and define $T = \inf\{t \in \mathbb{R}_+ : \langle M\rangle_t = n\}$. Then $M_{t\wedge T}$ is an Itô integral with a $\mathcal{L}^2$ integrand. Define $f(t) = \mathbb{E}(M_{t\wedge T}^2)$. By the Itô isometry

$$f(t) = \mathbb{E}\left(\int_0^t (A_s M_s + A_s N_0)^2 \mathbb{1}_{s\leq T}\mathrm{d}s\right)$$

Therefore for any $t' \in [0,t]$

$$f(t') \leq 2c(t)^2 N_0^2\, t' + 2c(t)^2 \int_0^{t'} f(s)\,\mathrm{d}s \leq \tilde{c}(t)\,t' + \tilde{c}(t)\int_0^{t'} f(s)\,\mathrm{d}s \tag{4.20}$$

where $\tilde{c}(t) = 2c(t)^2\max\{1, N_0^2\}$. This implies that $f(t') < \exp(2\tilde{c}(t)t')$ because no $t' \in [0,t]$ can be the smallest s such that $f(s) \geq \exp(2\tilde{c}(t)s)$ by (4.20). Therefore

$$\mathbb{E}(\langle N\rangle_{t\wedge T}) = \mathbb{E}(\langle M\rangle_{t\wedge T}) = f(t) < \exp(2\tilde{c}(t)t)$$

Taking $n \to \infty$ we get by monotone convergence that $\mathbb{E}(\langle N\rangle_t) \leq \exp(2\tilde{c}(t)t)$. This shows that the integrand $A_t N_t$ is in $\mathcal{L}^2$ and hence by the construction of the Itô integral, N_t is a martingale. □

Theorem 4.3.4. *Let $(p, r) \in \mathbb{R}^2$ be a solution of the equation*

$$r^2 - \left(1 + \frac{4}{\kappa}\right) r + \frac{2}{\kappa} p = 0$$

Then

$$\hat{M}_s = |\hat{h}'_s(z_0)|^p \, \hat{Y}_s^{p - \frac{\kappa}{2} r} \, (\sin \arg \hat{Z}_s)^{-2r}$$

is a martingale and

$$\mathbb{E}\left(|\hat{h}'_s(z_0)|^p \, (\sin \arg \hat{Z}_s)^{-2r} \right) = e^{-2s\left(p - \frac{\kappa}{2} r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Furthermore, if $r \geq 0$ and $p \geq 0$, then

$$\mathbb{P}\left(|\hat{h}'_s(z_0)| \geq \lambda \right) \leq \lambda^{-p} e^{-2s\left(p - \frac{\kappa}{2} r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Proof. We have already shown the first claim. For the second one notice that

$$\hat{M}_s = y_0^{p - \frac{\kappa}{2} r} \, e^{2s\left(p - \frac{\kappa}{2} r\right)} \, |\hat{h}'_s(z_0)|^p \, (\sin \arg \hat{Z}_s)^{-2r}.$$

If $r \geq 0$, then $(\sin \arg \hat{Z}_s)^{-2r} \geq 1$ and the last claim follows from the Chebyshev inequality. □

Corollary 4.3.5. *For every $0 \leq r \leq 1 + 4/\kappa$, there is $c = c(\kappa, r) < \infty$ such that for all $0 \leq t \leq 1$, $0 < y_0 \leq 1$, $e^6 \leq \lambda \leq y_0^{-1}$,*

$$\mathbb{P}\left(|\tilde{f}'_t(z_0)| \geq \lambda \right) \leq c \lambda^{-p} \left(\frac{y_0}{|z_0|}\right)^{-2r} \delta(y_0, \lambda). \tag{4.21}$$

Here $p = \frac{\kappa}{2}\left(\left(1 + \frac{4}{\kappa}\right) r - r^2\right) \geq 0$ and

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{-p + \frac{\kappa}{2} r} & \text{when } p - \frac{\kappa}{2} r > 0 \\ 1 + \log \frac{1}{\lambda y_0} & \text{when } p - \frac{\kappa}{2} r = 0 \\ y_0^{p - \frac{\kappa}{2} r} & \text{when } p - \frac{\kappa}{2} r < 0 \end{cases}.$$

Proof. Since $\tilde{f}'_t$ and h'_t have the same distribution, it is enough to show (4.21) when $\tilde{f}'_t$ is replaced by h'_t. Notice first that $Y_t \leq \sqrt{y_0^2 + 4t} \leq \sqrt{5}$. Therefore

$$\mathbb{P}\left(|h'_t(z_0)| \geq \lambda \right) \leq \mathbb{P}\left(\sup_{0 \leq s \leq T} |\hat{h}'_s(z_0)| \geq \lambda \right)$$

where $T = (\log(\sqrt{5}/y_0))/2$. Next notice that by (4.18), $|\hat{h}'_{s+s'}(z_0)| \leq e^{2s'} |\hat{h}'_s(z_0)|$ and therefore

$$\mathbb{P}\left(\sup_{0 \leq s \leq T} |\hat{h}'_s(z_0)| \geq \lambda \right) \leq \sum_{j=0}^{\lfloor T \rfloor} \mathbb{P}\left(|\hat{h}'_j(z_0)| \geq e^{-2} \lambda \right)$$

Also by (4.18), $|\hat{h}'_s(z_0)| \leq e^{2s}$ and therefore

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq s \leq T} |\hat{h}'_s(z_0)| \geq \lambda \right) &\leq \sum_{j = \lceil \log(\lambda)/2 - 1 \rceil}^{\lfloor T \rfloor} \mathbb{P}\left(|\hat{h}'_j(z_0)| \geq e^{-2} \lambda \right) \\ &\leq e^{2p} \lambda^{-p} \left(\frac{y_0}{|z_0|}\right)^{-2r} \sum_{j = \lceil \log(\lambda)/2 - 1 \rceil}^{\lfloor T \rfloor} e^{-2j\left(p - \frac{\kappa}{2} r\right)} \quad \leq c \, \lambda^{-p} \left(\frac{y_0}{|z_0|}\right)^{-2r} \delta(y_0, \lambda). \end{aligned}$$

Here we use that $\sum_{k=n}^{m} \beta^k \leq \beta^n / (1 - \beta)$ when $0 < \beta < 1$ and similar bounds for $\beta = 1$ and $\beta > 1$. □

Let's parametrize p in terms of r as

$$p(r) = \frac{\kappa}{2}\left(\left(1+\frac{4}{\kappa}\right)r - r^2\right)$$

and study the quantity

$$\alpha(r) = 2p(r) - \frac{\kappa}{2}r = \kappa\, r\left(\left(\frac{1}{2}+\frac{4}{\kappa}\right) - r\right).$$

Notice that $\alpha(r)$ is maximized by $r_0 = 1/4 + 2/\kappa$ and

$$\alpha(r_0) = \kappa\left(\frac{1}{4}+\frac{2}{\kappa}\right)^2 = \frac{\kappa}{16} + 1 + \frac{4}{\kappa} \geq 2$$

and $\alpha(r_0) = 2$ only if $\kappa = 8$.

Let $\kappa \neq 8$ and set $p_0 = p(r_0)$. Then $p_0 > \kappa r_0/2$ if $\kappa < 8$ and $p_0 < \kappa r_0/2$ if $\kappa > 8$. Let $\theta \in (0, 1 - \frac{2}{2p_0 - \kappa r_0/2})$. Let $t \in [0,1]$ and $n \in \mathbb{N}$. By the estimate (4.21) for $r = r_0$ and $p = p_0$ we have that for large enough n

$$\begin{aligned}\mathbb{P}\left(\left|\tilde{f}_t'(i2^{-n})\right| \geq 2^{n(1-\theta)}\right) &\leq c\, 2^{-p_0(1-\theta)n}\,\delta\left(2^{-n}, 2^{n(1-\theta)}\right)\\ &= c\, 2^{-p_0(1-\theta)n} \times \begin{cases} 2^{-(1-\theta)(p_0 - \frac{\kappa}{2}r_0)n} & \text{when } \kappa < 8\\ 2^{-(p_0-\frac{\kappa}{2}r_0)n} & \text{when } \kappa > 8\end{cases}\\ &\leq c\, 2^{-(1-\theta)(2p_0-\frac{\kappa}{2}r_0)n} = c2^{-(2+\varepsilon)n}\end{aligned}$$

for some $\varepsilon > 0$. Let

$$\mathcal{D}_{2n} = \{k\, 2^{-2n} \,:\, k = 0,1,2,\ldots,2^{2n}\}$$

which is the dyadic partitioning of $[0,1]$ into intervals of length 2^{-2n}. Then

$$\sum_{n\in\mathbb{N}}\sum_{t\in\mathcal{D}_{2n}} \mathbb{P}\left(\left|\tilde{f}_t'(i2^{-n})\right| \geq 2^{n(1-\theta)}\right) < \infty$$

and hence the Borel–Cantelli lemma implies the following result.

Proposition 4.3.6. *For each $\kappa \neq 8$ there exists $\theta_0(\kappa) > 0$ such that the following holds: For any $\theta \in (0, \theta_0(\kappa))$, there exists a random variable C such that $C < \infty$ almost surely and*

$$\left|\tilde{f}_t'(i2^{-n})\right| \leq C\, 2^{n(1-\theta)} \tag{4.22}$$

for any $t \in \mathcal{D}_{2n}$ and for any $n \in \mathbb{N}$.

Remark. By the above, we can choose

$$\theta_0(\kappa) = \frac{\frac{\kappa}{16}+\frac{4}{\kappa}-1}{\frac{\kappa}{16}+\frac{4}{\kappa}+1}.$$

4.3.3 The proof of Theorem 4.3.1

Proposition 4.3.7. *Let g_t be a Loewner chain with a driving term W_t. Suppose that*

$$\gamma(t) = \lim_{\varepsilon\searrow 0} g_t^{-1}(W_t + i\,\varepsilon)$$

exists for all $t \in \mathbb{R}_+$ and that $t \mapsto \gamma(t)$ is continuous. In that case, g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ and for all $t \in \mathbb{R}_+$, $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$.

We need the following properties about conformal maps: Let $g : U \to \mathbb{D}$ be a conformal onto map. Let $\alpha : [0,1) \to \mathbb{D}$ be a curve that extends continuously to its end point, $\overline{\alpha[0,1)} = (\alpha[0,1)) \cup \{z_0\}$ and $z_0 \in \partial U$. Then $\lim_{t\nearrow 1} g(\alpha(t)) = \zeta_0 \in \partial\mathbb{D}$ exists. This requires that the image domain (in this case $\mathbb{D}$) has locally connected boundary. Then the radial limit $\lim_{t\nearrow 1} g^{-1}(t\,\zeta_0)$ exists and equals to z_0. Therefore if there is another curve $\alpha' : [0,1) \to U$ such that $\lim_{t\nearrow 1}\alpha'(t)$ exists and $\lim_{t\nearrow 1} g(\alpha'(t)) = \zeta_0$ then $\lim_{t\nearrow 1}\alpha'(t) = z_0$

Proof. Let $S(t)$ be the set of all possible limit points of $g_t^{-1}(z)$ as $z \to W_t$ in $\mathbb{H}$. Then $S(t) \subset \partial H_t$ by Theorem 2.2.5 and $S(t)$ is non-empty because $\gamma(t) \in S(t)$. Fix $t_0 \in \mathbb{R}_+$ and let $z_0 \in S(t_0)$. We will show that $z_0 \in \overline{\gamma[0,t_0)}$. Let $\varepsilon > 0$ and let

$$t_\varepsilon = \sup \left\{ t \in \mathbb{R}_+ \,:\, K_t \cap \overline{B(z_0,\varepsilon)} = \emptyset \right\}$$

Then $B(z_0,\varepsilon) \cap H_{t_0} \neq \emptyset$ and we can choose $z \in B(z_0,\varepsilon) \cap H_{t_0}$. Also $\overline{B(z_0,\varepsilon)} \cap K_{t_\varepsilon} \neq \emptyset$ and we can choose $z' \in \overline{B(z_0,\varepsilon)} \cap K_{t_\varepsilon}$. Let z'' to be the first point in K_{t_ε} in the line segment from z to z' and let α be the line segment $[z,z'')$. Then $g_{t_\varepsilon} \circ \alpha$ extends continuously to its end point which is some $x \in \mathbb{R}$. Suppose that $x \neq W_{t_\varepsilon}$. Then $g_t(z'')$ has to hit W_t for some $t < t_\varepsilon$ which contradicts with the definition of t_ε. Hence $x = W_{t_\varepsilon}$. Now $\gamma(t_\varepsilon) = \lim_{\delta\searrow 0} g_{t_\varepsilon}^{-1}(W_{t_\varepsilon} + i\,\delta)$ has to be equal to z''. Therefore we have shown that $z'' \in \gamma[0,t_0)$ and that $\gamma[0,t_0) \cap \overline{B(z_0,\varepsilon)} \neq \emptyset$. Since $\varepsilon > 0$ is arbitrary, $z_0 \in \overline{\gamma[0,t_0)} = \gamma[0,t_0]$ and $\bigcup_{t \le t_0} S(t) = \gamma[0,t_0]$.

We will next show that $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. First note that H_t is connected and disjoint from $\gamma[0,t] = \bigcup_{t' \le t} S(t') \subset \bigcup_{t' \le t} \partial H_{t'}$. By the same argument as in the previous paragraph, $(\mathbb{H} \cap \partial H_t) \subset \gamma[0,t]$ and therefore H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. Since the boundary of H_t is locally connected g_t^{-1} extends continuously to $\overline{\mathbb{H}}$. □

We will use the results of the following exercisees.

Exercise. Show using the Koebe distortion theorem that there exists constants C and r such that for any conformal map $f : \mathbb{H} \to \mathbb{C}$ and for any $x \in \mathbb{R}$, $y > 0$ and $1/2 \le s \le 2$

$$C^{-1}|f'(iy)| \le |f'(isy)| \le C|f'(iy)| \tag{4.23}$$

$$C^{-1}(1+x^2)^{-r}|f'(iy)| \le |f'(y(x+i))| \le C(1+x^2)^r|f'(iy)|. \tag{4.24}$$

What is the value of r that you get from the Koebe distortion theorem?

Exercise. (a) Let g_t be a Loewner chain and $f_t = g_t^{-1}$. By differentiating the Loewner equation of f_t with respect to z, find a differential equation for $f_t'(z)$. Show that for $x \in \mathbb{R}$, $y > 0$

$$|\partial_t f_t'(x+iy)| \le \frac{2|f_t''(x+iy)|}{y} + \frac{2|f_t'(x+iy)|}{y^2}.$$

(b) Show using the special case $|a_2| \le 2$ of the Bieberbach–de Branges theorem that there is a constant $c > 0$ such that

$$|f''(z)| \le \frac{c}{\operatorname{Im} z}|f'(z)|$$

for any $f : \mathbb{H} \to \mathbb{C}$ conformal and for any $z \in \mathbb{H}$.

(c) Show that there are constants c_1, c_2, c_3 such that following holds for any Loewner chain: for any $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y > 0$

$$|\partial_t f_t'(x+iy)| \le \frac{c_1|f_t'(x+iy)|}{y^2}$$

and if $0 \le s \le y^2$ then

$$c_2^{-1}\,|f_t'(x+iy)| \le |f_{t+s}'(x+iy)| \le c_2\,|f_t'(x+iy)| \tag{4.25}$$

$$|f_{t+s}(x+iy) - f_t(x+iy)| \le c_3\,y\,|f_t'(x+iy)|. \tag{4.26}$$

Definition 4.3.8. An increasing, continuous function $\psi : [0,\infty) \to (0,\infty)$ is said to be *subpower function* if

$$\lim_{x\to\infty} \frac{\log \psi(x)}{\log x} = 0.$$

or equivalently if for all $\mu > 0$

$$\lim_{x\to\infty} x^{-\mu}\psi(x) = 0.$$

Remark. One way to write this is $\psi(x) = \exp(o(\log x))$. If ψ_1 and ψ_2 are subpower functions also $\psi_1\psi_2$, $\psi_1 + \psi_2$ and $\psi(x) = \psi_1(x^p)$, $p > 0$, are subpower functions.

Proof of Theorem 4.3.1. By the previous proposition it is enough to prove that the trace exists and is continuous. Our goal is to prove this based on the following bounds: As we saw above for each $\kappa \neq 8$, there exist a constant $\theta > 0$ and a random variable C which is almost surely finite such that

$$|\tilde{f}_t'(i2^{-n})| \leq C\,2^{n(1-\theta)} \tag{4.27}$$

for all $t \in \mathcal{D}_{2n}$ and for any $n \in \mathbb{N}$. Remember also that since $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion, there is an almost surely finite random variable $\tilde{C}$ such that

$$|W_{t+s} - W_t| \leq \tilde{C}\sqrt{s\,\log(1/s)} \tag{4.28}$$

for any $t, s \in [0, 1]$.

Fix a realization of the driving process and the Loewner chain such that the bounds (4.27) and (4.28) hold for some finite C and $\tilde{C}$. Throughout this proof let c be a generic constant that might change from line to line and let ψ be a generic subpower function. We allow that c and ψ can depend on C and $\tilde{C}$ but they don't depend on the other variables such as t and y.

Let $t \in [0, 1], y \in (0, 1)$. Take $n \in \mathbb{N}$ and $t_0 \in \mathcal{D}_{2n}$ such that

$$2^{-n} \leq y < 2^{-n+1}, \qquad t_0 \leq t < t_0 + 2^{-2n},$$

that is, $n = \lceil \log_2(1/y) \rceil$ and $t_0 = \lfloor t2^{2n} \rfloor 2^{-2n}$. By (4.27) and (4.28)

$$\begin{aligned} |\tilde{f}_t'(iy)| &= |f_t'(W_t + iy)| \leq c|f_{t_0}'(W_t + iy)| \\ &\leq c|f_{t_0}'(W_t + iy_0)| \leq c\left(1 + \frac{|W_t - W_{t_0}|^2}{y_0^2}\right)^r |f_{t_0}'(W_{t_0} + iy)| \\ &\leq cn^r 2^{n(1-\theta)} \leq y^{\theta-1}\psi(1/y) \end{aligned} \tag{4.29}$$

for some subpower function ψ.

Let's integrate the bound (4.29). By change of integration variable

$$\int_0^y u^{\theta-1}\psi(1/u)\,\mathrm{d}u = y^\theta\tilde{\psi}(1/y)$$

where

$$\tilde{\psi}(x) = \int_0^1 u^{\theta-1}\psi(x/u)\,\mathrm{d}u.$$

It is not difficult to check that $\tilde{\psi}$ is a subpower function. Hence

$$\gamma(t) = \lim_{y\searrow 0} \tilde{f}_t(iy)$$

exists and satisfies

$$|\gamma(t) - \tilde{f}_t(iy)| \leq y^\theta\psi(1/y). \tag{4.30}$$

We still have to show that $t \mapsto \gamma(t)$ is continuous. It is enough to estimate

$$|\gamma(t+s) - \gamma(t)|$$

when $t \in \mathcal{D}_{2n}$, $0 \leq s \leq 2^{-2n}$. By triangle inequality

$$|\gamma(t+s) - \gamma(t)| \leq |\gamma(t+s) - \tilde{f}_{t+s}(iy)| + |\tilde{f}_{t+s}(iy) - \tilde{f}_t(iy)| + |\tilde{f}_t(iy) - \gamma(t)|$$

Set $y = 2^{-n}$. Then by (4.30)

$$|\gamma(t+s) - \gamma(t)| \leq 2^{-n\theta}\psi(2^n) + |\tilde{f}_{t+s}(i2^{-n}) - \tilde{f}_t(i2^{-n})|$$

Again by triangle inequality

$$|\tilde{f}_{t+s}(i2^{-n}) - \tilde{f}_t(i2^{-n})| \leq \underbrace{|f_{t+s}(W_{t+s} + i2^{-n}) - f_{t+s}(W_t + i2^{-n})|}_{=A} + \underbrace{|f_{t+s}(W_t + i2^{-n}) - f_t(W_t + i2^{-n})|}_{=B}.$$

By (4.29) and (4.24)

$$|f'_{t+s}(x+i2^{-n})| \le c(1+2^{2n}(x-W_{t+s})^2)^r |f'_{t+s}(W_{t+s}+i2^{-n})| \le 2^{n(1-\theta)}\psi(2^n) \tag{4.31}$$

when x lies on the interval between W_t and W_{t+s}. Integrating this bound over the interval between W_t and W_{t+s} gives

$$A \le 2^{n(1-\theta)}\psi(2^n)\cdot \tilde{C}2^{-n}\sqrt{n} = 2^{-n\theta}\psi(2^n)$$

By (4.26)

$$B \le c2^{-n}|f'_t(W_t+i2^{-n})| \le c\,2^{-\theta n}$$

We have now shown that

$$|\gamma(t+s)-\gamma(t)| \le 2^{-n\theta}\psi(2^n)$$

for $t\in\mathcal{D}_{2n}$, $0\le s\le 2^{-2n}$. For $0<s\le 2^{-2n}$, we can choose $m\ge n$ such that $2^{-2m-2}<s\le 2^{-2m}$ and since $\mathcal{D}_{2n}\subset\mathcal{D}_{2m}$ when $n\le m$, we can apply the previous bound to show that

$$|\gamma(t+s)-\gamma(t)| \le s^{\frac{\theta}{2}}\psi(1/s) \tag{4.32}$$

for $t\in\mathcal{D}_{2n}$, $0<s\le 2^{-2n}$. □

In fact, the bound (4.32) implies the following result.

Proposition 4.3.9. *For each $\kappa\neq 8$, there exist a constant $\alpha_0>0$ such that $t\mapsto\gamma(t)$ is Hölder continuous for any exponent $\alpha<\alpha_0$*

Remark. We can choose $\alpha_0=\theta_0/2$ where θ_0 is as in the remark after Proposition 4.3.6.

4.4 Coordinate change of SLE(κ) and simple consequences

Let's summarize here some fact from the exercises. Let $c<0$. The unique map ϕ_c from $\mathbb{H}$ onto the strip $S_\pi=\{z\in\mathbb{C}\,:\,0<\operatorname{Im}z<\pi\}$ with $\phi_c(0)=0$, $\phi_c(c)=-\infty$ and $\phi_c(\infty)=+\infty$ is

$$\phi_c(z)=\log(z-c)-\log|c|.$$

Let g_t be the Loewner chain in $\mathbb{H}$ associated to a driving term W_t. Then

$$\hat{G}_t=\log\left[g_t\left(c+|c|e^z\right)-g_t\left(c\right)\right]-\frac{1}{2}\log g'_t(c)-\log|c|.$$

defines a map from $S_\pi\setminus\phi_c(K_t)$ onto S_π and $\hat{G}_t$ is the unique such map with the expansion near $\pm\infty$ equal to

$$\hat{G}_t(z)=\begin{cases} z-S(t)+o(1), & z\to-\infty\\ z+S(t)+o(1), & z\to+\infty\end{cases}.$$

Especially $S(t)$ is uniquely determined by these requirements and in this case

$$S(t)=-\frac{1}{2}\log g'_t(c)=\int_0^t\frac{\mathrm{d}u}{(W_t-g_t(c))^2}.$$

Now the driving term is transformed to

$$\hat{W}_t=\log(W_t-g_t(c))+S(t)-\log|c|\ \in\mathbb{R}\subset\partial S_\pi.$$

Define a time-change $\sigma(s)$ such that $S(\sigma(s))=s$ and set $G_s=\hat{G}_{\sigma(s)}$ and $\tilde{W}_s=\hat{W}_{\sigma(s)}$. A straightforward calculation shows that G_s satisfies the Loewner equation of the strip S_π

$$\partial_s G_s(z)=\coth\frac{G_s(z)-\tilde{W}_s}{2},\qquad G_0(z)=z$$

And if the driving term in the upper half-plane is a Brownian motion then the driving term of the strip is a Brownian motion with a drift. See Table 4.1 for more details.

	$\mathbb{H}$	S_π
Normalization	$g_t(z) = z + \frac{2t}{z} + \ldots,\ z \to \infty$	$G_s(z) = \begin{cases} z - s + o(1), & z \to -\infty \\ z + s + o(1), & z \to +\infty \end{cases}$
Loewner equation	$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$	$\partial_s G_s(z) = \coth \frac{G_s(z) - \tilde{W}_s}{2}$
Driving term	$W_t \in \mathbb{R}$	$\tilde{W}_s \in \mathbb{R}$
Chordal SLE(κ)	$W_t = \sqrt{\kappa} B_t$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha_0(\kappa) s$
SLE(κ, ρ)	$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$
Relations between parameters	$\alpha = \rho + 3 - \frac{\kappa}{2}, \qquad \alpha_0(\kappa) = 3 - \frac{\kappa}{2}$	

Table 4.1: A comparison between SLE in $\mathbb{H}$ and in S_π.

4.4.1 Definition of SLE(κ, ρ)

Also in the exercises we saw that the transformation from S_π to $\mathbb{H}$ sends driving processes of the form $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$ to a process W_t which is as in the next definition.

Definition 4.4.1. Let $\kappa \geq 0$ and $\rho \in \mathbb{R}$. Let $w_0, c_0 \in \mathbb{R}$ with $w_0 \neq c_0$ and let $(W_t, C_t)_{t \in [0, \tau(c_0))}$ be the solution to the system of stochastic differential equations

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}, \qquad \begin{cases} W_0 = w_0 \\ C_0 = c_0 \end{cases} \tag{4.33}$$

which exists for $t \in [0, \tau(c_0))$ where $\tau(c_0) = \sup\{t \in \mathbb{R}_+ : \inf_{s \in [0,t]} |W_t - C_t| > 0\}$. Then the Loewner chain $(g_t, K_t)_{t \in [0, \tau(c_0))}$ with the driving process $(W_t)_{t \in [0, \tau(c_0))}$ is called *SLE*(κ, ρ).

Remark. The chordal SLE(κ) is a special case SLE($\kappa, 0$) of this definition.

It is possible to construct SLE(κ, ρ) using a Bessel process. This construction is especially useful when we want to consider the process beyond $\tau(c_0)$, which we don't do in these lecture notes, but we'll give this construction here. Let $w_0, c_0 \in \mathbb{R}$ with $w_0 \neq c_0$ and let $\eta = \mathrm{sgn}(w_0 - c_0)$. Let D_t be the Bessel process (with unusual time-parametrization)

$$\mathrm{d}D_t = \frac{\rho + 2}{D_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}\tilde{B}_t, \qquad D_0 = |w_0 - c_0|.$$

Define

$$C_t = c_0 - 2\eta \int_0^t \frac{\mathrm{d}u}{D_u}, \qquad W_t = C_t + \eta D_t.$$

Then they satisfy (4.33) with $B_t = \eta \tilde{B}_t$.

4.4.2 Schramm's principle and SLE(κ, ρ)

We will now generalize Schramm's principle which was given in the section 4.1.1. Assume that we are given a collection of probability measures $(\mathbb{P}^{U,a,b,c})$ indexed by the set all triplets (U, a, b, c) where U is any simply connected domain and $a \neq b \neq c \neq a$ are boundary points of U in counterclockwise order. Assume that $\mathbb{P}^{U,a,b,c}$ is the law of a random curve $\gamma : [0, \infty) \to \mathbb{C}$ (the parametrization is arbitrary) such that $\gamma([0, \infty)) \subset \overline{U}$ and $\gamma(0) = a$. We assume that the family $(\mathbb{P}^{(U,a,b,c)})$ satisfies the following properties:

- The family $(\mathbb{P}^{(U,a,b,c)})$ satisfies **conformal invariance** (CI):

$$\phi_* \mathbb{P}^{(U,a,b,c)} = \mathbb{P}^{(\phi(U), \phi(a), \phi(b), \phi(c))}$$

- The family $(\mathbb{P}^{(U,a,b,c)})$ satisfies **domain Markov property** (DMP):

$$\mathbb{P}^{(U,a,b,c)}(\gamma|_{[t,\infty)} \in B \mid \mathcal{F}_t) = \mathbb{P}^{(U\setminus\gamma([0,t]),\gamma(t),b,c)}(\gamma \in B)$$

  for any measurable set B in the space of curves.

We also assume that we can describe the curve γ by the Loewner equation in the sense that $\mathbb{P}^{(S_\pi,0,+\infty,-\infty)}$ is supported on curves which have a continuous driving term in the Loewner equation of S_π. Then a similar calculation as in the section 4.1.1 will show that the driving process is a Brownian motion with drift

$$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s \tag{4.34}$$

where $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. In the section 4.1.1, we used scale invariance to deduce that there is no drift in the driving process. Now we don't have any extra symmetries left. Therefore the Loewner chain of the strip satisfies Schramm's principle for three points if and only if the driving process is of the form (4.34).

4.4.3 SLE$(\kappa, (\kappa - 6)/2)$

Denote the reflection with respect to the y-axis by

$$m(z) = -\overline{z}. \tag{4.35}$$

Then m is an antiholomorphic map from $\mathbb{C}$ onto itself. Since the process $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s$ is invariant under $\tilde{W}_s \mapsto -\tilde{W}_s$, SLE$(\kappa, (\kappa-6)/2)$ on S_π is invariant under m and for fixed $\kappa > 0$, it is the unique SLE(κ, ρ) process with this property. We say that SLE$(\kappa, (\kappa-6)/2)$ on S_π is *symmetric*.

Suppose that we know that some discrete random curve arising from statistical physics converges to SLE(κ) as the mesh goes to zero. For example, suppose we know that the interface of Ising model with boundary conditions changing at two marked points (boundary conditions are + spins on one arc and − spins on the other arc) converges to SLE(3). Can we conclude something about the scaling limit for other boundary conditions? If we consider the Ising model with three marked points $a, b, c \in \partial U$ (in counterclockwise order) and boundary conditions are set to be − on the arc ab, + on the arc ca and free on the arc bc, then by Schramm's principle we expect that the scaling limit of the interface starting from the point a should be SLE$(3, \rho)$ process. And since the law of that interface is invariant under flipping all the spins $\sigma \to -\sigma$, the scaling limit should be symmetric on S_π and hence it should be SLE$(3, -3/2)$.

4.4.4 Locality of SLE(6)

Consider following map

$$\psi = m \circ \phi^{-1} \circ m \circ \phi \tag{4.36}$$

where m is as in (4.35). Under those maps SLE(κ) is transformed as

$$\begin{aligned}(\mathbb{H}, \kappa = 6, \rho = 0) &\xrightarrow{\phi} (S_\pi, \kappa = 6, \alpha = 0) \xrightarrow{m} (S_\pi, \kappa = 6, \alpha = 0) \\ &\xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa = 6, \rho = 0) \xrightarrow{m} (\mathbb{H}, \kappa = 6, \rho = 0)\,.\end{aligned}$$

On the other hand ψ is a holomorphic and bijective self map of $\mathbb{H}$ with $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\psi(c) = |c|$. Hence

$$\psi(z) = \frac{|c|z}{z - c}. \tag{4.37}$$

Therefore SLE(6) has the following *locality* property: the image of SLE(6) under any conformal self-map of $\mathbb{H}$ is again (a time-change of) SLE(6). If $\psi : \mathbb{H} \to \mathbb{H}$ is this Möbius map, then we consider the first process until it disconnects $\psi^{-1}(\infty)$ from ∞ and the second one until it disconnects $\psi(\infty)$ from ∞. Actually SLE(6) has even stronger locality property because SLE(6) sent from 0 is invariant up to a time-change under any conformal transformation defined in a neighborhood of 0 such that it maps the real axis around 0 in $\mathbb{R}$.

4.4.5 SLE$(\kappa, \kappa - 6)$

For other values of κ, the argument of the section 4.4.4 gives that if $(K_t)_{t\in[0,\tau(c))}$ is a chordal SLE(κ) stopped at the time $\tau(c)$ then $(\psi(K_t))_{t\in[0,\tau(c))}$ is a time-change of the SLE$(\kappa, \kappa-6)$ process stopped at the time when the process disconnects $|c|$ from ∞. Namely, under the map ψ of the form (4.36) the processes are transformed in the following way:

$$(\mathbb{H}, \kappa, \rho = 0) \xrightarrow{\phi} (S_\pi, \kappa, \alpha = 3 - \kappa/2) \xrightarrow{m} (S_\pi, \kappa, \alpha = \kappa/2 - 3)$$
$$\xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa, \rho = \kappa - 6) \xrightarrow{m} (\mathbb{H}, \kappa, \rho = \kappa - 6)\,.$$

4.5 Dimension of SLE

Let $K \subset \mathbb{C}$ be a non-empty bounded Borel set. Let N_ε be the number of sets of the form

$$[(j-1)\varepsilon, j\varepsilon] \times [(k-1)\varepsilon, k\varepsilon], \quad (j,k) \in \mathbb{Z}^2$$

intersecting with the set K. Then the box-counting dimension (or Minkowski dimension) of K is defined to be

$$\dim_{\mathrm{M}}(K) = \lim_{\varepsilon\searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}}$$

if the limit exists. If the limit doesn't exists we define the upper and lower box-counting dimensions as

$$\dim_{\overline{\mathrm{M}}}(K) = \limsup_{\varepsilon\searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}}, \qquad \dim_{\underline{\mathrm{M}}}(K) = \liminf_{\varepsilon\searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}},$$

respectively. It is true that the (upper and lower) box-counting dimension is not less than the Hausdorff dimension of K. Hence any upper bound for the box-counting dimension is an upper bound for the Hausdorff dimension. In this section we will show that the following upper bound for SLE(κ), $0 < \kappa < 8$, curve γ:

$$\dim_{\overline{\mathrm{M}}}(([-1,1] \times [0,1]) \cap \gamma[0,\infty)) \le 1 + \frac{\kappa}{8}. \tag{4.38}$$

This bound is sharp. In fact, it is possible to show that the Hausdorff dimension of SLE(κ) is almost surely $1 + \kappa/8$. Remember that for any $z_0 \in \overline{\mathbb{H}}$, $\mathbb{P}(z_0 \in \gamma[0,\infty)) = 1$. Therefore almost surely $\gamma[0,\infty) = \overline{\mathbb{H}}$ and the Hausdorff dimension of $\gamma[0,\infty)$ is 2 almost surely, when $\kappa \ge 8$.

Suppose that for some $C > 0$ and $\lambda > 0$, we have a bound

$$\mathbb{P}\left(\gamma[0,\infty) \cap \overline{B(z_0, r)} \neq \emptyset\right) \le C \left(\frac{r}{\operatorname{Im} z_0}\right)^\lambda \tag{4.39}$$

for all $z_0 \in \mathbb{H}$ and $r > 0$. If $\gamma[0,\infty)$ intersects $R_{j,k} = [(j-1)\,2^{-n}, j\,2^{-n}] \times [(k-1)\,2^{-n}, k\,2^{-n}]$, then

$$\operatorname{dist}\left(\left(j - \frac{1}{2}\right) 2^{-n} + i \left(k - \frac{1}{2}\right) 2^{-n}, \gamma[0,\infty)\right) \le 2^{-n-1/2}.$$

Hence

$$\begin{aligned} \mathbb{E}N_{2^{-n}} &= \sum_{\substack{-2^n<j\le 2^n \\ 0<k\le 2^n}} \mathbb{P}\left(\gamma[0,\infty) \cap R_{j,k} \neq \emptyset\right) \le C\,2^{-\lambda/2} \sum_{\substack{-2^n<j\le 2^n \\ 0<k\le 2^n}} (k - 1/2)^{-\lambda} \\ &\le C' 2^{(2-\lambda)n} \end{aligned}$$

where C' is a constant that depends only on C and λ.

Now by Chebyshev inequality, for each $\delta > 0$

$$\mathbb{P}\left(N_{2^{-n}} \ge 2^{(2-\lambda+\delta)n}\right) \le C' 2^{-\delta n}.$$

Since these probabilities are summable over n, by Borel–Cantelli lemma there exist a random $n_0(\delta)$ such that

$$N_{2^{-n}} < 2^{(2-\lambda+\delta)n} \tag{4.40}$$

for $n > n_0(\delta)$. Now

$$\limsup_{n\to\infty} \frac{\log N_{2^{-n}}}{n \log 2} \le 2 - \lambda + \delta$$

Since $\delta > 0$ is arbitrary and $\varepsilon \mapsto N_\varepsilon$ is increasing

$$\limsup_{\varepsilon\searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}} \le 2 - \lambda.$$

Hence the upper box-counting dimension is at most $2 - \lambda$. Therefore to obtain the claim (4.38), we have to show (4.39) with $\lambda = \lambda(\kappa) = 1 - \kappa/8$, for $0 < \kappa < 8$.

To show (4.39), we first need a conformally invariant version of the distance to the boundary. This quantity is known as *conformal radius.*

Exercise. Let $U \subset \mathbb{C}$ be a simply connected domain with $U \neq \mathbb{C}$ and let $z_0 \in U$. Let ψ be the unique conformal map from U onto $\mathbb{D}$ such that $\psi(z_0) = 0$ and $\psi'(z_0) > 0$. Then the *conformal radius of U from z_0* is defined as

$$\rho(z_0, U) = \frac{1}{\psi'(z_0)}.$$

(a) Show that if ϕ is a conformal map from U onto $\mathbb{D}$ with $\phi(z_0) = 0$ then $\rho(z_0, U) = |\phi'(z_0)|^{-1}$. Show also that $\rho(\lambda z_0, \lambda U) = \lambda \rho(z_0, U)$ for $\lambda > 0$ and $\rho(f(z_0), f(U)) = |f'(z_0)|\rho(z_0, U)$ for any conformal map $f : U \to \mathbb{C}$.

(b) Show using the Koebe distortion theorem that

$$\operatorname{dist}(z_0, \partial U) \le \rho(z_0, U) \le 4 \operatorname{dist}(z_0, \partial U).$$

(c) Let g be a conformal map from U onto $\mathbb{H}$. Show that

$$\rho(z_0, U) = \frac{2 \operatorname{Im} g(z_0)}{|g'(z_0)|}.$$

By the previous exercise, the conformal radius of $H_t = \mathbb{H} \setminus K_t$ from z_0 is

$$\rho(z_0, H_t) = \frac{2Y_t}{|g_t'(z_0)|}$$

when $t < \tau(z_0)$. And

$$\frac{1}{2} \operatorname{dist}(z_0, \partial H_t) \le \frac{Y_t}{|g_t'(z_0)|} \le 2 \operatorname{dist}(z_0, \partial H_t).$$

This implies that

$$\frac{1}{2} \operatorname{dist}(z_0, \gamma[0,\infty)) \le \lim_{t\nearrow\tau(z_0)} \frac{Y_t}{|g_t'(z_0)|} \le 2 \operatorname{dist}(z_0, \gamma[0,\infty)).$$

Define a time-change

$$S(t) = \int_0^t \frac{2 \sin\theta_u \, \mathrm{d}u}{\sqrt{X_u^2 + Y_u^2}}, \qquad \sigma(s) = S^{-1}(s).$$

We use the definition

$$\hat{\theta}_s = 2\theta_{\sigma(s)}$$

since it corresponds to the coordinate change from $\mathbb{H}$ to $\mathbb{D}$. Now

$$\mathrm{d}\hat{\theta}_s = \frac{\kappa - 4}{2} \cot\left(\frac{\hat{\theta}_s}{2}\right) \mathrm{d}s + \sqrt{\kappa}\mathrm{d}\hat{B}_s \tag{4.41}$$

$$\rho(z_0, H_{\sigma(s)}) = 2y_0 e^{-s} \tag{4.42}$$

We also see that

$$\lim_{t\nearrow\tau(z_0)} \frac{Y_t}{|g_t'(z_0)|} = \rho(z_0, U_0) = 2y_0 e^{-\hat{\tau}(z_0)}$$

where $\hat{\tau}(z_0) = S(\tau(z_0))$ and U_0 is the connected component of z_0 in $\mathbb{H} \setminus \gamma[0, \infty)$.

For $z_0 = r \exp(i\hat{\theta}_0/2)$, $r > 0$, define a function, which doesn't depend on $r > 0$,

$$F(\hat{\theta}_0, u) = \mathbb{P}\left(\rho(z_0, U_0) \le 2y_0 e^{-u}\right) = \mathbb{P}(\hat{\tau}(z_0) \ge u).$$

A conditional version of this is when $s < T_{\{0,2\pi\}} = \sup\{s \in \mathbb{R}_+ : \inf_{s' \in [0,s]} \hat{\theta}_s > 0 \text{ and } \sup_{s' \in [0,s]} \hat{\theta}_s < 2\pi\}$

$$\mathbb{P}\left(\hat{\tau}(z_0) \ge u \,\middle|\, \hat{F}_s\right) = F(\hat{\theta}_s, u - s)$$

by the conformal Markov property of SLE(κ) or by Markov property of the pair of processes (4.41) and (4.42). This is by construction a martingale and therefore F satisfies

$$\dot{F} = LF \tag{4.43}$$

where L is the following second order differential operator

$$Lf(x) = -\frac{\kappa}{2} f''(x) - \frac{\kappa - 4}{2} \cot \frac{x}{2} f'(x).$$

The function F satisfies the boundary conditions

$$F(x, 0) = 1, \quad 0 < x < 2\pi \qquad \text{and} \qquad F(0, u) = 0 = F(2\pi, u), \quad u > 0. \tag{4.44}$$

To analyze the asymptotic behaviour of the solution of (4.43) and (4.44) we need to find a positive eigenfuntion of L. Namely, in a suitable function space L is a self-adjoint operator. Moreover, there exists a eigenbasis $(f_k)_{k \in \mathbb{N}}$ of L such that $Lf_k = \lambda_k f_k$, $0 < \lambda_1 < \lambda_2 < \ldots$ and for each k, f_k has $k - 1$ zeros on the interval $(0, 2\pi)$. It is straightforward to check that

$$f(x) = \sin\left(\frac{x}{2}\right)^{\beta}$$

satisfies $Lf = \lambda f$ if and only if

$$\lambda = 1 - \frac{\kappa}{8}, \qquad \beta = \frac{8}{\kappa} - 1.$$

Since the boundary values (4.44) are non-negative, it is possible to prove the following version of the maximum principle: there exists a constant $C > 0$ such that

$$C^{-1} f(x) e^{-\lambda u} \le F(x, u) \le C f(x) e^{-\lambda u}$$

for all $x \in [0, 2\pi]$ and $u \ge 1$. Since $f \le 1$, (4.39) follows.

The bound (4.39) can be used to show the following result which is the final piece of the theory of "phases of SLE".

Proposition 4.5.1. *When $0 < \kappa < 8$, for each $z \in \overline{\mathbb{H}}$, $\mathbb{P}(z \in \gamma[0, \infty)) = 0$. Furthermore, when $4 < \kappa < 8$, for each $z \in \mathbb{H}$, $\mathbb{P}(z$ is swallowed$) = 1$.*