Chapter 3

Loewner chains

3.1 Conformal maps of the upper-half plane

Definition 3.1.1. A set $K \subset \overline{\mathbb{H}}$ is called a *hull* if K is compact and $\mathbb{H} \setminus K$ is simply connected.

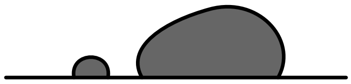


Figure 3.1: A hull K (the shaded area in the picture) is a compact subset of the closed upper half-plane $\overline{\mathbb{H}}$ such that its complement $\mathbb{H} \setminus K$ is simply connected. Especially, each point in K is connected to the real axis within K.

Remark. An alternative definition of a hull is the following: $K \subset \mathbb{H}$ is a hull if K is bounded, relatively closed in $\mathbb{H}$ and $\mathbb{H} \setminus K$ is simply connected. For any hull K in this sense, $\overline{K}$ is a hull in the above sense and for any hull K in this sense of Definition 3.1.1, $K \cap \mathbb{H}$ is a hull in the sense of the latter definition.

Lemma 3.1.2. *For any hull K, there exists a unique conformal and onto map $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that*

$$\lim_{z \to \infty} (g_K(z) - z) = 0 \tag{3.1}$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ such that $|z_n| \to \infty$. Such g_K is said to have hydrodynamic normalization. *Near ∞, g_K has the expansion*

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots$$

where the coefficients a_k, $k \in \mathbb{N}$, are real.

Proof. If $\tilde{g} : \mathbb{H} \setminus K \to \mathbb{D}$ is a conformal onto map, then $\tilde{g}(\infty) \in \partial\mathbb{D}$ is well-defined since there is a holomorphic extension of $z \mapsto \tilde{g}(-1/z)$ to a neighborhood of 0 by Theorem 2.2.6. Hence we can compose $\tilde{g}$ with a Möbius map from $\mathbb{D}$ onto $\mathbb{H}$ mapping $\tilde{g}(\infty)$ to ∞ and get this way a conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ mapping ∞ to ∞. By this observation and by the Riemann mapping theorem, there are conformal onto maps from $H = \mathbb{H} \setminus K$ onto $\mathbb{H}$ which map ∞ to ∞. Pick one of them and call it $\hat{g}$. Let $H' = \{-1/z : z \in H\}$ and

$$f(z) = -1/\hat{g}(-1/z). \tag{3.2}$$

By Theorem 2.2.6, f extends holomorphically and injectively to a neighborhood of 0. Let $\varepsilon > 0$ be such that $B(0, \varepsilon) \cap \mathbb{H} \subset H'$. Then f maps $(-\varepsilon, \varepsilon)$ into $\mathbb{R}$. Moreover, if $f = u + i\,v$, then $f'(0) = \partial_x u(0) = \partial_y v(0) > 0$ because f maps $B(0, \varepsilon) \cap \mathbb{H}$ into $\mathbb{H}$. Hence

$$f(z) = b_1 z + b_2 z^2 + \dots$$

near 0 where the coefficients satisfy $b_1 > 0$ and $b_j \in \mathbb{R}$. For $\hat{g}$ this implies that for large $|z|$

$$\hat{g}(z) = \hat{a}_{-1} z + \hat{a}_0 + \hat{a}_1 z^{-1} + \hat{a}_2 z^{-2} + \dots \tag{3.3}$$

where the coefficients satisfy $\hat{a}_{-1} > 0$ and $\hat{a}_j \in \mathbb{R}$. Now we notice that $\hat{g}$ satisfies (3.1), if and only if $\hat{a}_{-1} = 1$ and $\hat{a}_0 = 0$.

As stated in the remark after the Riemann mapping theorem (Theorem 2.2.4), if $g : H \mapsto \mathbb{H}$ is a conformal onto map taking ∞ to ∞, then all the other such maps can be written as $\phi \circ g$ where ϕ is a Möbius self-map of $\mathbb{H}$ fixing ∞. The Möbius self-maps of $\mathbb{H}$ that fix ∞ are of the form

$$z \mapsto \alpha z + \beta$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. Hence for given $\hat{g}$ there is a unique choice for ϕ such that $g_K = \phi \circ \hat{g}$ has the expansion

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots$$

for $z \in \mathbb{H} \setminus B(0, R)$. □

Lemma 3.1.3. *The coefficient a_1 is non-negative and $a_1 = 0$ only if g_K is an identity map.*

Proof. Define a harmonic function h in $\mathbb{H} \setminus K$ by

$$h(z) = \operatorname{Im}(z - g_K(z)).$$

Then the boundary values of h are non-negative: it is zero on $\mathbb{R}$ away from K and on $\partial K \cap \mathbb{H}$ it is positive. Also $h(z) \to 0$ as $|z| \to \infty$. Hence by the minimum principle, h is non-negative in $\mathbb{H} \setminus K$. In fact, h is strictly positive unless $h = 0$ identically and g_K is an identity map. Now

$$\lim_{y \nearrow \infty} y\, h(i\, y) = \lim_{y \nearrow \infty} y \operatorname{Im}\left(-\frac{a_1}{i\, y} + \mathcal{O}\left(|y|^{-2}\right) \right) = a_1$$

which shows that $a_1 \geq 0$. The strict positivity follows when we notice that

$$a_1 = \frac{2R}{\pi} \int_0^\pi h(R\, e^{i\theta}) \sin\theta \,\mathrm{d}\theta. \tag{3.4}$$

That formula follows from the previous formula and from the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \{z \in \mathbb{H} : |z| > R\} \\ u(x) = 0 & \text{for } x \in \mathbb{R}, |x| \geq R \\ u(Re^{i\theta}) = \phi(\theta) & \text{for } \theta \in (0, \pi) \end{cases}$$

in terms of a Poisson kernel. The proof of the formula (3.4) is left as an exercise. □

Definition 3.1.4. If K is a hull and g_K satisfies the hydrodynamic normalization, then the coefficient of z^{-1} in the expansion of g_K is denoted by $a_1(K)$. We call $a_1(K)$ as the *half-plane capacity* of K.

The half-plane capacity satisfies the following properties:

- Scaling rule: $a_1(\lambda K) = \lambda^2 a_1(K)$ because

$$g_{\lambda K}(z) = \lambda g_K(\lambda^{-1} z) = z + \lambda^2 a_1(K) z^{-1} + \dots$$

- Summation rule: $a_1(K \cup L) = a_1(K) + a_1(g_K(L))$. Let $L' = g_K(L)$. Then

$$g_{K \cup L}(z) = g_{L'} \circ g_K(z) = z + (a_1(K) + a_1(L'))z^{-1} + \dots$$

- Translation invariance: $a_1(K + x) = a_1(K)$

$$g_{K+x}(z) = x + g_K(z - x) = z + a_1(K) z^{-1} + \dots$$

From the summation rule and from Lemma 3.1.3 it follows that if $J \subset K$ are hulls then $a_1(J) \leq a_1(K)$ and $a_1(J) = a_1(K)$ only if $\mathbb{H} \cap (K \setminus J) = \emptyset$. We say that *half-plane capacity is increasing.* These properties make the half-plane capacity very natural measure for the size of the hull K (as seen from ∞).

Example 3.1.5. When $K = \overline{\mathbb{H} \setminus B(x_0, R)}$, then

$$g_K(z) = z + \frac{R^2}{z - x_0} = z + \frac{R^2}{z} + \frac{R^2 x_0}{z^2} + \dots$$

because by a direct computation $g_K(x) \in \mathbb{R}$ when $x \in \mathbb{R}$, $|x - x_0| \geq R$, and

$$g_K(x_0 + Re^{i\theta}) = x_0 + 2R\cos\theta \in \mathbb{R}.$$

The half-plane capacity is now $a_1(K) = R^2$.

Example 3.1.6. $K = [x_0, x_0 + i\,\delta] = \{x_0 + i\,y \,:\, y \in [0, \delta]\}$

$$\begin{aligned} g_K(z) &= x_0 + \sqrt{(z - x_0)^2 + \delta^2} \\ &= x_0 + z\sqrt{1 - \frac{2x_0}{z} + \frac{x_0^2 + \delta^2}{z^2}} \\ &= x_0 + z\left(1 - \frac{x_0}{z} + \frac{x_0^2 + \delta^2}{2z^2} - \frac{1}{8}\frac{4x_0^2}{z^2} + \dots\right) \\ &= z + \frac{\delta^2}{2z} + \dots \end{aligned}$$

where we used the expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$ The half-plane capacity is now $a_1(K) = \delta^2/2$.

We conclude this section by showing that the half-plane capacity is a continuous function of the hull. We present also two very useful auxiliary results needed in the proof. For a hull K and $\varepsilon > 0$, let K^ε be the ε-*thickening* of K, that is, K^ε is the smallest hull containing the set $\mathbb{H} \cap \bigcup_{z \in K} \overline{B(z, \varepsilon)}$.

Lemma 3.1.7. *There are constants $C(R) > 0$ and $\alpha > 0$ such that the following holds: If $K \subset K^\varepsilon \subset B(z_0, R)$, then*

$$a_1(K) \leq a_1(K^\varepsilon) \leq a_1(K) + C(R)\varepsilon^\alpha$$

Proof. The inequality on the left follows from the summation rule and the positivity of the half-plane capacity.

To show the other inequality consider the harmonic functions $h_K(z) = \operatorname{Im}(z - g_K(z))$ and $h_{K^\varepsilon}(z) = \operatorname{Im}(z - g_{K^\varepsilon}(z))$. Note that they are both non-negative and bounded by R and note also that they are continuous in $\overline{\mathbb{H} \setminus K}$ and $\overline{\mathbb{H} \setminus K^\varepsilon}$, respectively.

Let $z \in \mathbb{H} \cap \partial K^\varepsilon$. Then $\operatorname{dist}(z, K) = \varepsilon$. Let $\mathbb{P}^z$ be the law of a complex Brownian motion send from z and let τ be the hitting time of $\mathbb{R} \cup K$. Then by Lemma 3.1.8 below

$$h_K(z) = \mathbb{E}^z(\operatorname{Im} B_\tau)$$

and by definition $h_{K^\varepsilon}(z) = \operatorname{Im} z$. Write

$$\begin{aligned} |h_K(z) - h_{K^\varepsilon}(z)| &\leq \mathbb{E}^z(|\operatorname{Im} B_\tau - \operatorname{Im} z|) \\ &= \mathbb{E}^z(|\operatorname{Im} B_\tau - \operatorname{Im} z| \,;\, \sigma < \tau) + \mathbb{E}^z(|\operatorname{Im} B_\tau - \operatorname{Im} z| \,;\, \sigma = \tau) \end{aligned} \tag{3.5}$$

where is σ be the exit time from $(\mathbb{H} \setminus K) \cap B(z, \sqrt{\varepsilon})$. The first term on the right of (3.5) is at most $R\,\mathbb{P}^z(\sigma < \tau)$ and hence by Lemma 3.1.8 below, there are constants $\tilde{\alpha} > 0$ and $\tilde{C} > 0$ such that the first term is bounded by $\tilde{C}R(\varepsilon/\sqrt{\varepsilon})^{\tilde{\alpha}} = \tilde{C}R\varepsilon^{\tilde{\alpha}/2}$. The second term is at most $\sqrt{\varepsilon}$.

Now since for some constants $C(R) > 0$ and α, $|h_K(z) - h_{K^\varepsilon}(z)| \leq C(R)\varepsilon^\alpha$ on the boundary of $\mathbb{H} \setminus K^\varepsilon$ and $h_K - h_{K^\varepsilon}$ is a bounded harmonic function on $\mathbb{H} \setminus K^\varepsilon$, the maximum principle gives that $|h_K(z) - h_{K^\varepsilon}(z)| \leq C(R)\varepsilon^\alpha$ on $\mathbb{H} \setminus K^\varepsilon$. Therefore the formula (3.4) can be applied to show that $|a_1(K) - a_1(K^\varepsilon| \leq C(R)\varepsilon^\alpha$ □

Lemma 3.1.8. *Let U be a domain and $h : \overline{U} \to \mathbb{R}$ be a bounded continuous function such that h is harmonic in U. Let $z \in U$ and $B_t^{(z)}$ be a complex Brownian motion send from z. Assume that $\tau_z = \inf\{t \in \mathbb{R}_+ \,:\, B_t^{(z)} \notin U\}$ is almost surely finite. Then $h(B_{t \wedge \tau_z}^{(z)})$ is a bounded continuous martingale and*

$$h(z) = \mathbb{E}\left(h\left(B_{\tau_z}^{(z)}\right)\right).$$

Proof. The fact that $M_t = h(B^{(z)}_{t\wedge\tau_z})$ is a local martingale follows from Itô's formula as we saw in the proof of the conformal invariance of Brownian motion. Since h is bounded, M_t is a bounded continuous martingale and we can use optional stopping to show that $M_0 = \mathbb{E}(M_{\tau_z})$. □

Lemma 3.1.9 (Weak Beurling estimate)**.** *There exist constant $\alpha > 0$ and $C > 0$ such that the following holds: Let $D = \mathbb{D} \setminus \gamma[0, 1)$ where $\gamma : [0, 1) \to \mathbb{D}$ be a simple curve with $\gamma(0) = 0$ and $\lim_{t\nearrow 1} |\gamma(t)| = 1$. Let $\mathbb{P}^z$ be the law of complex Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ send from $z \in D$ and let τ be its exit time from D. Then for any $z \in D$*

$$\mathbb{P}^z(|B_\tau| = 1) \le C|z|^\alpha$$

Remark. The result is called weak since the proof below only gives that there is some exponent $\alpha > 0$. It doesn't give the optimal exponent which is $\alpha = 1/2$.

Proof. Consider a complex brownian motion send from $w \in \mathbb{C}$ with $|w| = 2$ and let $\sigma = \inf\{t \in \mathbb{R}_+ : |B_t| = 1 \text{ or } 4\}$. By rotational invariance of the complex Brownian motion

$$q = \mathbb{P}^w(B([0, \tau]) \text{ contains a loop around } 0)$$

is independent of w. Now $q > 0$ follows from a more general fact that the probability that d-dimensional Brownian motion follows any given continuous path with a given precision up to a given time is positive.

Let $\rho = |z|$. Now if $|B_\tau| = 1$ then the Brownian motion $B_t, 0 \le t \le \tau$, will hit the circles of radii $r_k = \rho\, 2^k$, $k = 0, 1, 2, \ldots, n_0(\rho)$ centered at 0 where $n_0(\rho)$ is the largest integer n such that $\rho\, 2^n \le 1$. Denote the hitting times of those circles by T_k, $k = 0, 1, \ldots, n_0(\rho)$. If for some $k = 0, 1, \ldots, n_0(\rho) - 1$, B_t, $t \ge T_k$, makes a loop around 0 before hitting the circles of radii r_{k-1} or r_{k+1}, then the Brownian motion hits ∂D and $|B_\tau| < 1$. Apply the strong Markov property for T_k, $k = 0, 1, \ldots, n_(\rho) - 1$, to show that

$$\mathbb{P}^z(|B_\tau| = 1) \le (1 - q)^{n_0(\rho)}.$$

Then note that $n_0(\rho) > (\log(1/\rho))/(\log 2) - 1$ and hence the claim holds for $C = 1/(1 - q)$ and $\alpha = (\log 1/(1 - q))/(\log 2)$. □

3.1.1 Growing families of hulls

Let I be an interval of the form $[0, \infty), [0, T]$ or $[0, T)$ where $T \in (0, \infty)$. Let $\gamma : I \to \overline{\mathbb{H}}$ be curve such that $\gamma(0) \in \mathbb{R}$. We can define a family of hulls $(K_t)_{t\in I}$ associated to $\gamma(t)$, $t \in I$, in the following way:

- If γ is simple (a curve is simple if and only if it is injective) and $\gamma(t) \subset \mathbb{H}$, $t > 0$, then define $K_t = \gamma([0, t])$ for any $t \in I$.
- If γ is not simple let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$ and let $K_t = \overline{\mathbb{H} \setminus H_t}$.

If γ is simple both of the above definition would give the same hulls $(K_t)_{t\in I}$.

Let $(K_t)_{t\in I}$ be a family of hulls parametrized by a real variable $t \in I$ where I is as above. The family of hulls associated to a curve is a good example of such family. If the family $(K_t)_{t\in I}$ is *growing* in the sense that $K_s \subset K_t$ for $s \le t$ and if *the growth is continuous* in the sense that for any $\varepsilon > 0$ and for any $S \in (0, \infty)$ such that $[0, S] \subset I$ there exist $\delta > 0$ such that $K_{t+\delta} \subset K_t^\varepsilon$ for any $0 \le t \le S - \delta$, then by Lemmas 3.1.3 and 3.1.7, the function $\phi : t \mapsto a_1(K_t)$ is continuous and non-decreasing. If we assume that $K_0 \subset \mathbb{R}$ and that $\mathbb{H} \cap (K_t \setminus K_s) \ne \emptyset$ for any $0 \le s < t \le T$, then $\phi(0) = 0$ and by the summation rule and by the positivity of the half-plane capacity $\phi(t) > \phi(s)$ for any $0 \le s < t \le T$. Hence we can reparametrize the family of hulls by setting $\tilde{K}_t = K_{\phi^{-1}(2t)}$. In this parametrization $a_1(\tilde{K}_t) = \phi(\phi^{-1}(2t)) = 2t$. This can be summarized by saying that continuously growing families of hulls can be parametrized by capacity.

Definition 3.1.10. A family of hulls $(K_t)_{t\in[0,T)}$ is said to be *parametrized with the half-plane capacity* if $a_1(K_t) = 2t$. A curve $\gamma : [0, T) \to \overline{\mathbb{H}}$ is said to be *parametrized with the half-plane capacity* if the associated hulls are parametrized with the half-plane capacity.

For a given family of hulls $(K_t)_{t\in I}$ it is convenient to set

$$g_t = g_{K_t}.$$

If $(K_t)_{t\in[0,T)}$ is parametrized by the capacity then

$$g_t(z) = z + \frac{2t}{z} + \dots$$

From now on we assume (almost without exceptions) that g_t is a conformal map with this form of an expansion near ∞. Often it is useful to call the parameter t as time. The factor 2 is because of historical reasons: using that normalization the Loewner equation in $\mathbb{H}$ will be better compatible with the Loewner equation in $\mathbb{D}$, as we will later see. And we chose that the half-plane capacity is linear in t because of the summation rule.

3.2 Loewner chains

3.2.1 Loewner equation holds for simple curves

Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls parametrized with the half-plane capacity. Let

$$\tilde{K}_{t,s} = \overline{g_t(K_{t+s} \setminus K_t)}, \qquad \tilde{g}_{t,s} = g_{\tilde{K}_{t,s}}. \tag{3.6}$$

The reason to choose the parametrization with the half-plane capacity is that $a_1(\tilde{K}_{t,s}) = 2s$, that is, if we remove the hull K_t from the hull K_{t+s} by applying the conformal map g_t, then the resulting hull will have half-plane capacity $2(t+s) - 2t = 2s$.

The idea in the Loewner equation is to consider for fixed z the flow $t \mapsto g_t(z)$. We can apply the expansion near ∞ to the difference

$$g_{t+\delta}(z) - g_t(z) = \tilde{g}_{t,\delta} \circ g_t(z) - g_t(z) = \frac{2\delta}{z} + \dots$$

Therefore we expect that $\partial_t g_t(z)$ is non-trivial if it exists and that it should have at least one singularity somewhere in the complex plane. If we consider a family of hulls, which is growing *locally*, that is, from a single point, then we expect that there should be a singularity at the point $P_t \in \partial K_t$ where the hull is growing at time t. It turns out, as we will soon see, that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}$$

where $W(t) = g_t(P_t)$ when we suitably interpret what is the value of g_t at a boundary point P_t.

In this section, we present the result that the conformal maps g_t associated to a simple curve γ satisfy the Loewner equation.

Theorem 3.2.1. *Let $T > 0$ and let $\gamma : [0,T] \to \mathbb{C}$ be a simple curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(\,(0,T]\,) \subset \mathbb{H}$. Suppose that γ is parametrized by the capacity. Then*

$$W(t) = \lim_{z\to\gamma(t)} g_t(z) \tag{3.7}$$

exists for any $t \in [0,T]$ and $t \mapsto W(t)$ is continuous. Here the limit is along any sequence $z_n \in \mathbb{H}\setminus\gamma(0,t]$ converging to $\gamma(t)$. Moreover the hydrodynamically normalized conformal maps $(g_t)_{t\in[0,T]}$ related to γ satisfy the Loewner differential equation (of the upper half-plane)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{3.8}$$

with the initial value $g_0(z) = z$.

Before the proof of this theorem we present three auxiliary results.

Lemma 3.2.2. *Let K be a hull and $H = \mathbb{H} \setminus K$. If $K \subset B(x_0, r)$, then g_K maps $H \cap B(x_0, 2r)$ into $B(x_0, 3r)$ and $\sup_{z \in H} |g_K(z) - z| \le 5r$.*

Proof. We can assume that $x_0 = 0$. Otherwise consider the map $g_{K-x_0}(z) = g_K(z + x_0) - x_0$.

Let $\tilde{g}$ be the holomorphic extension of $r^{-1} g_K(r\, z)$ to $\mathbb{D}^*$. Then $\tilde{g} \in \Sigma$ and by the Area theorem $\sum_{n=1}^{\infty} n |a_n(K)|^2 r^{-2(n+1)} \le 1$ and therefore $|a_n(K)| \le r^{n+1}$. Hence

$$|g_K(z) - z| \le \sum_{n=1}^{\infty} |a_n(K)|\, |z|^{-n} \le r \sum_{n=1}^{\infty} (r/|z|)^n = \frac{r^2}{|z| - r} \le r$$

for $|z| \ge 2r$. And therefore $g_K(\mathbb{H} \cap B(0, 2r)) \subset B(0, 3r)$.

If $z \in H \cap B(0, 2r)$, then $|g_K(z) - z| \le |g_K(z)| + |z| < 5r$. □

Using the next lemma we can control the length distortion under conformal maps. This lemma could be used in the proof of the general result Theorem 2.2.7 about the continuity of conformal maps to the boundary. The same principle of proof could be systematized by introducing so called *extremal length*.

Lemma 3.2.3. *Let ϕ be a conformal map from open set $U \subset \mathbb{C}$ into $B(0, R)$. Let $z_0 \in \mathbb{C}$ and let $C(r) = U \cap \{z : |z - z_0| = r\}$ for any $r > 0$. Then*

$$\inf_{\rho < r < \sqrt{\rho}} \{\mathrm{Length}\,(\phi(\, C(r)\,))\} \le \frac{2\pi R}{\sqrt{\log 1/\rho}}. \tag{3.9}$$

Proof. Let $l(r) = \mathrm{Length}\,(\phi(\, C(r)\,))$. By the Cauchy–Schwarz inequality

$$\begin{aligned} l(r)^2 &= \left(\int_{C(r)} |\phi'(z)|\, |\mathrm{d}z| \right)^2 \le \int_{C(r)} |\mathrm{d}z| \int_{C(r)} |\phi'(z)|^2\, |\mathrm{d}z| \\ &\le 2\pi r \int_{z_0 + re^{i\theta} \in U} \left|\phi'\left(z_0 + re^{i\theta}\right)\right|^2 r\, \mathrm{d}\theta. \end{aligned}$$

Divide this by r and then integrate over r to find that

$$\int_0^{\infty} l(r)^2\, r^{-1}\, \mathrm{d}r \le 2\pi \int_U |\phi'(z)|^2\, \mathrm{d}x\mathrm{d}y = 2\pi \mathrm{Area}(\phi(U))$$

which implies that

$$\frac{1}{2} \log \frac{1}{\rho} \left(\inf_{\rho < r < \sqrt{\rho}} l(r)^2 \right) \le \int_{\rho}^{\sqrt{\rho}} l(r)^2\, r^{-1}\, \mathrm{d}r \le 2\pi^2 R^2.$$

The claim follows by taking a square root. □

Lemma 3.2.4. *There exist an absolute constant $C > 0$ such that the following holds: If $K \subset B(x_0, r) \cap \mathbb{H}$ and $z \in \mathbb{H}$, $|z - x_0| \ge Cr$, then*

$$\left| f_K(z) - z + \frac{a_1(K)}{z - x_0} \right| \le \frac{C r a_1(K)}{|z - x_0|^2}$$

where $f_K = g_K^{-1}$

Proof. We can assume $x_0 = 0$. We can also assume that the boundary of K is a continuous. If not, then take a sequence K_n each having a continuous boundary and such that $f_{K_n} \to f_K$ uniformly in compact subsets of $\mathbb{H}$.

A similar argument as above shows that f_K has an expansion of the form (3.3) and a direct calculation then tells that

$$f_K(z) = z - a_1 z^{-1} + \ldots$$

Let

$$h(z) = \mathrm{Im}(f_K(z) - z).$$

Then h is a bounded continuous function in $\overline{\mathbb{H}}$ and harmonic in $\mathbb{H}$. Hence we can write h using the Poisson kernel of $\mathbb{H}$ as

$$h(z) = \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} h(\xi) \, \mathrm{d}\xi.$$

We can use this formula to derive the harmonic conjugate of h. Notice $h = \operatorname{Im} f_K$ on $\mathbb{R}$ and therefore

$$f_K(z) = z + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \operatorname{Im} f_K(\xi) \, \mathrm{d}\xi.$$

The additive constant in the harmonic conjugate of h was fixed by the expansion near ∞.

Now clearly $\operatorname{Im} f_K(\xi)$ is zero outside a bounded interval I which is defined as the smallest interval containing $\{\xi \in \mathbb{R} : f_K(\xi) \in \mathbb{H} \cap \partial K\}$. From this it follows that

$$\begin{aligned} f_K(z) &= z + \frac{1}{\pi} \int_I \frac{1}{\xi - z} \operatorname{Im} f_K(\xi) \, \mathrm{d}\xi \\ &= z - \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_I \xi^{n-1} \operatorname{Im} f_K(\xi) \, \mathrm{d}\xi \right) z^{-n} \end{aligned}$$

for large enough $|z|$. Hence

$$a_1 = \frac{1}{\pi} \int_I \operatorname{Im} f_K(\xi) \, \mathrm{d}\xi$$

and

$$\begin{aligned} \left| f_K(z) - z + \frac{a_1(K)}{z} \right| &= \left| \frac{1}{\pi} \int_I \left(\frac{1}{\xi - z} + \frac{1}{z} \right) \operatorname{Im} f_K(\xi) \, \mathrm{d}\xi \right| \\ &\leq a_1(K) \, \sup \left\{ \left| \frac{1}{x - z} + \frac{1}{z} \right| \, : \, x \in I \right\} \end{aligned}$$

By Lemma 3.2.2, $I \subset (-3r, 3r)$ and hence

$$\left| \frac{x}{(x - z) z} \right| \leq \frac{6r}{|z|^2}$$

for any $|z| \geq 6r$ and $x \in I$. □

Proof of Theorem 3.2.1. As usual, denote $H_t = \mathbb{H} \setminus \gamma(0, t]$. Since $\gamma[0, T]$ is bounded, we can define $R = \sup_{t \in [0,T]} |\gamma(t)| < \infty$.

For each $t \in [0, T]$ and $r > 0$, let $S(t, r)$ be the outermost of all the connected components of $H_t \cap \partial B(\gamma(t), r)$ which separate $\gamma(t)$ from ∞ in H_t. See Figure 3.2. Since by Lemma 3.2.2, g_t maps $H_t \cap B(0, 2R)$ into $\underline{B(0, 3R)}$, we can apply Lemma 3.2.3 to g_t and show that the diameter of $g_t(S(t, r))$ is at most $6\pi R / \sqrt{\log(1/r)}$. Since the curves $g_t(S(t, r))$, $r > 0$ are nested (in the sense that for any $0 < r_1 < r_2$, $g_t(S(t, r_2))$ separates $g_t(S(t, r_1))$ from ∞ in $\mathbb{H}$) and their diameters go to zero as $r \searrow 0$, there exist $W(t) \in \mathbb{R}$ such that

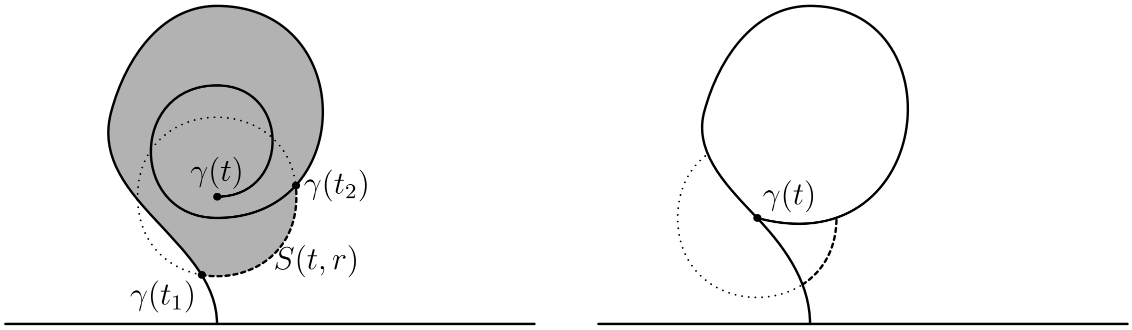
$$\{W(t)\} = \bigcap_{r > 0} \overline{V(t, r)}$$

where $V(t, r)$ is the bounded component of $\mathbb{H} \setminus g_t(S(t, r))$.

Since γ is simple, $g_t(H_t \cap B(\gamma(t), r')) \subset V(r, t)$ for small enough $r' > 0$. Namely, when $r < \operatorname{Im} \gamma(t)$, the end points of $S(t, r)$ are points $\gamma(t_1), \gamma(t_2)$, for some $0 < t_1 < t_2 < t$. Since the distance from $\gamma(t)$ to $\gamma([t_1, t_2]) \cup S(t, r)$ is positive, then for small enough $r' > 0$, $S(t, r)$ separates $H_t \cap B(\gamma(t), r')$ from ∞ in H_t. See Figure 3.2(a). Therefore $g_t(H_t \cap B(\gamma(t), r')) \subset V(r, t)$ and

$$\{W(t)\} = \bigcap_{r' > 0} \overline{g_t(H_t \cap B(\gamma(t), r'))}.$$

Hence $g_t(\gamma(t)) = \lim_{z \to \gamma(t)} g_t(z)$ is well-defined and the first claim follows.



(a) The arc $S(t,r)$ of the circle of radius r centered at $\gamma(t)$ separates the tip $\gamma(t)$ from ∞ in H_t and it is outermost of all such arcs (in the sense that it separates all other such arcs from ∞ in H_t). When $\gamma(t) \in \mathbb{H}$ and r is small, the end points of $S(t,r)$ lie on the curve.

(b) When the curve is not simple, the conformal map doesn't extend continuously to the tip $\gamma(t)$ when that point is a double point visible from more than one side of the curve. The correct solution for this problem is to define the corresponding generalized boundary point as a nested sequence of arcs of circles. These generalized boundary points are called *prime ends*.

Figure 3.2: Continuity of the conformal map g_t at the tip point $\gamma(t)$ follows from the fact that an arc $S(t,r)$ of a small circle is mapped to a set of small diameter.

Now for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $t \in [0, T-\delta]$ we have that $g_t(\gamma(t, t+\delta]) \subset V(t,\varepsilon)$. Denote the conformal map associated to the hull $g_t(\gamma([t, t+\delta]))$ by $\tilde{g}_{t,\delta}$ and let $\tilde{\gamma}(\delta) = g_t(\gamma(t+\delta))$. Since $\operatorname{diam} V(t,\varepsilon) \le r_0(\varepsilon) = 6\pi R/\sqrt{\log(1/\varepsilon)}$, $\tilde{\gamma}(\delta) \in \mathbb{H} \cap B(W(t), r_0(\varepsilon))$ and therefore by Lemma 3.2.2

$$|W(t+\delta) - W(t)| = |\tilde{g}_{t,\delta}(\tilde{\gamma}(\delta)) - W(t)| \le 3 r_0(\varepsilon). \tag{3.10}$$

Therefore $t \mapsto W(t)$ is continuous.

Let $C > 0$ be as in Lemma 3.2.4. Let $t \in [0,T)$, $z \in H_t$ and choose $\varepsilon > 0$ so small that $(C+5) r_0(\varepsilon) < \operatorname{Im} g_t(z)$. If $0 < \delta \le T - t$ is such that $g_t(\gamma(t, t+\delta]) \subset V(t,\varepsilon)$ then

$$\begin{aligned} |g_{t+\delta}(z) - W(t)| &\ge |g_t(z) - W(t)| - |\tilde{g}_{t,\delta} \circ g_t(z) - g_t(z)| \\ &\ge C r_0(\varepsilon). \end{aligned}$$

Use Lemma 3.2.4 for the map $f_K = \tilde{g}_{t,\delta}^{-1}$ at point $g_{t+\delta}(z)$ with $r = r_0(\varepsilon)$ and $x_0 = W(t)$ to find that

$$\left| g_{t+\delta}(z) - g_t(z) - \frac{2\delta}{g_{t+\delta}(z) - W(t)} \right| \le \frac{2\delta C r_0(\varepsilon)}{|g_{t+\delta}(z) - W(t)|^2}.$$

Since we can take $r_0(\varepsilon) \searrow 0$ as $\delta \searrow 0$, the derivative from the right exists and satisfies

$$\partial_{t+}\, g_t(z) = \lim_{\delta \searrow 0} \frac{g_{t+\delta}(z) - g_t(z)}{\delta} = \frac{2}{g_t(z) - W(t)}$$

Since the right-hand side is continuous in t, actually, $\partial_t g_t(z)$ exists and we have shown that (3.8) holds. □

Example 3.2.5. Let $\delta(t) = 2\sqrt{t}$ and let $g_t(z) = \sqrt{z^2 + \delta(t)^2}$. Then

$$\partial_t g_t(z) = \frac{2}{g_t(z)}.$$

Therefore the driving term of the straight vertical line $t \mapsto i\delta(t)$, $t \ge 0$, is $W_t = 0$ for all t.

3.2.2 Solving Loewner equation with a continuous driving term

In this section, we study the solution of Loewner equation with a continuous driving term and show that there is a growing family of hulls parametrized with the half-plane capacity. In fact, we will show that there is one-to-one correspondence between *locally growing hulls* and the solutions of the Loewner equation with continuous driving terms.

Let $t \mapsto W_t$ be a given real valued function on $[0,T]$. In this section we will investigate whether there is a family of conformal maps $(g_t)_{t\in[0,T]}$ that satisfy the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z. \tag{3.11}$$

First fix $z \in \overline{\mathbb{H}}$. Then (3.11) is just an ordinary differential equation (ODE) in the parameter t. Furthermore, the mapping

$$\zeta \mapsto \frac{2}{\zeta - W_t} \tag{3.12}$$

is continuous in t and Lipschitz continuous in ζ in the set of points

$$\{(t,\zeta) \in [0,T] \times \overline{\mathbb{H}} \,:\, |\zeta - W_t| \geq \varepsilon\}$$

where $\varepsilon > 0$. Therefore by the theory of ODEs there exists a unique solution to

$$\dot{z}_t = \frac{2}{z_t - W_t}, \qquad z_0 = z \tag{3.13}$$

for the whole interval $[0,T]$, if $|z_t - W_t|$ remains positive for all $t \in [0,T]$, otherwise the solution exists for $t \in [0,\tau(z))$ where $\tau(z)$ is the smallest time $u \in [0,T]$ such that $\inf_{0\leq t<u} |z_t - W_t| = 0$. If the solution exists for the whole $[0,T]$, set $\tau(z) = \infty$. Now set

$$g_t(z) = z_t$$

for $t \in [0,T] \cap [0,\tau(z))$ and we claim that this defines a conformal map.

Define the domain of g_t as

$$H_t = \{z \in \mathbb{H} \,:\, \tau(z) > t\}.$$

By continuity in t and by Lipschitz continuity in ζ of (3.12), $z \mapsto g_t(z)$ is a continuous map and H_t is an open set. Namely, if $g_t(z)$ is well-defined then the solution of (3.13) for any initial point in a small neighborhood of z is well-defined at least up to time t and that solution remains close to the solution for z. We leave this as an exercise and formulate below this as a lemma .

Since (3.12) and the initial condition are both holomorphic, the solution $g_t(z)$ will be holomorphic in z. To see this explicitly, let $z, z' \in \mathbb{H}$ and let

$$D_t(z,z') = g_t(z) - g_t(z')$$

for any $t \in [0,T] \cap [0,\tau(z) \wedge \tau(z'))$. It satisfies the differential equation

$$\dot{D}_t(z,z') = -D_t(z,z')\,\frac{2}{(g_t(z) - W_t)(g_t(z') - W_t)}$$

and therefore

$$D_t(z,z') = (z - z')\exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z) - W_s)(g_s(z') - W_s)}\right). \tag{3.14}$$

Hence the complex derivative $g_t'(z)$ exists and equals to

$$g_t'(z) = \lim_{z'\to z} \frac{D_t(z,z')}{z - z'} = \exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z) - W_s)^2}\right).$$

This shows that g_t is holomorphic. In addition, (3.14) shows that g_t is one-to-one. Therefore $g_t : H_t \to \mathbb{C}$ is a conformal map.

We will show that $g_t(H_t) = \mathbb{H}$. Note first, that

$$\partial_t \operatorname{Im} g_t(z) = -2\,\frac{\operatorname{Im} g_t(z)}{|g_t(z) - W_t|^2}$$

and hence $\operatorname{Im} g_t(z)$ is strictly decreasing and positive by the formula

$$\operatorname{Im} g_t(z) = (\operatorname{Im} z) \exp\left(- \int_0^t \frac{2\mathrm{d}s}{|g_s(z) - W_s|^2} \right)$$

which holds for any $t \in [0,T] \cap [0, \tau(z))$. Therefore $g_t(H_t) \subset \mathbb{H}$. Fix $t \in (0,T]$ and let $w \in \mathbb{H}$. Define $h_s(w)$ as the solution of the *backward Loewner equation*

$$\partial_s h_s(w) = -\frac{2}{h_s(w) - W_{t-s}}, \qquad h_0(w) = w. \tag{3.15}$$

Then $h_s(w)$, $0 \le s \le t$, is well-defined and lies in the upper half-plane, because $\operatorname{Im} h_s(w)$ is strictly increasing. Let $z = h_t(w)$. Then then $g_s(z) = h_{t-s}(w)$ because $s \mapsto h_{t-s}(w)$ solves the (forward) Loewner equation with the initial condition $h_t(w) = z$. Escpecially $g_t(z) = w$ and we have shown that $g_t(H_t) = \mathbb{H}$.

Set now

$$K_t = \{z \in \overline{\mathbb{H}} \,:\, \tau(z) \le t\}. \tag{3.16}$$

Then $H_t = \mathbb{H} \backslash K_t$. We will show that K_t is a hull. Obviously $K_t \subset \overline{\mathbb{H}}$, it is closed and its complement is simply connected. To show that K_t is bounded let $M = \sup_{t \in [0,T]} |W_t|$. For any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z > M$, $\operatorname{Re} g_s(z)$ is strictly increasing since

$$\partial_s \operatorname{Re} g_s(z) = 2 \frac{\operatorname{Re}(g_s(z)) - W_s}{|g_s(z) - W_s|^2} > 0$$

when $\operatorname{Re} g_s(z) > M$. Similarly for any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z < -M$, $\operatorname{Re} g_s(z)$ is strictly decreasing. For any $z \in \mathbb{H}$ with $\operatorname{Im} z > 2\sqrt{t}$

$$\partial_s \operatorname{Im} g_s(z) = -2 \frac{\operatorname{Im}(g_s(z))}{|g_s(z) - W_s|^2} \ge -\frac{2}{\operatorname{Im}(g_s(z))}$$

and hence

$$(\operatorname{Im} g_t(z))^2 \ge (\operatorname{Im} z)^2 - 4t > 0.$$

Therefore

$$\left\{ z \in \mathbb{H} \,:\, |\operatorname{Re} z| > M \text{ or } \operatorname{Im} z > 2\sqrt{T} \right\} \subset H_t \tag{3.17}$$

and

$$K_t \subset \left\{ z \in \overline{\mathbb{H}} \,:\, |\operatorname{Re} z| \le M \text{ and } \operatorname{Im} z \le 2\sqrt{T} \right\}. \tag{3.18}$$

Now we have established that g_t is a conformal map from $H_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ and that K_t is a hull. Note from the above considerations concerning the real and imaginary parts of g_t we can deduce that for fixed $t \mapsto U_t$, $0 \le t \le T$, $g_t(z) = z + o(1)$ as $|z| \to \infty$. We can apply Lemma 3.1.2 to show that g_t has the expansion

$$g_t(z) = z + \sum_{k=1}^{\infty} a_k(t) z^{-k}$$

which converges uniformly for $|z| > R$ where $R > 0$ satisfies $K_t \subset \overline{B(0,R) \cap \mathbb{H}}$. Hence

$$\partial_t g_t(z) = \frac{2}{z} + \dots$$

and hence $a_1(t) = 2t$.

Based on the above, we formulate two results. First is about continuity of the solution of (3.13) as function of the initial value and the driving term.

Lemma 3.2.6. *Let $T > 0$. For each $\varepsilon > 0$, there exists $\delta_0, \delta_1 > 0$ such that the following holds. If $z_0 \in \mathbb{H}$, $W \in C([0,T])$ and z_t is the solution of the equation*

$$\dot{z}_t(z) = \frac{2}{z_t - W_t}, \ t \in [0,T]$$

and they satisfy the condition

$$\inf_{t\in[0,T]} |z_t - W_t| \geq \varepsilon,$$

then for any $\tilde{z}_0 \in B(z_0, \delta_0)$ *and* $\tilde{W} \in C([0,T])$ *with* $\|W - \tilde{W}\|_\infty < \delta_1$ *the solution of*

$$\dot{\tilde{z}}_t(z) = \frac{2}{\tilde{z}_t - \tilde{W}_t}$$

exists for all $t \in [0,T]$ *and it satisfies the condition*

$$\inf_{t\in[0,T]} |\tilde{z}_t - \tilde{W}_t| \geq \varepsilon/2.$$

Furthermore, the inequality

$$|z_t - \tilde{z}_t| \leq e^{At}\left(|z_0 - \tilde{z}_0| + A^{-1}\left(1 - e^{-At}\right)\|W^{(n)} - W\|_{\infty,[0,T]}\right)$$

holds where $A = 4\varepsilon^{-2}$.

Proof. We leave the proof as an exercise. $\square$

The following theorem will give an equivalent condition to the fact that g_t has a continuous driving term. This condition is called *local growth*. The result generalizes Theorem 3.2.1. See Figure 3.3 for some examples related to the theorem.

Theorem 3.2.7. *Let* $(K_t)_{t\in[0,T]}$ *be a growing family of hulls and* g_t *be the associated conformal maps. Then the following statements are equivalent:*

- *For all* $t \in [0,T]$, $a_1(K_t) = 2t$ *and for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for each* $t \in [0, T-\delta]$, *there exists a bounded connected set* $C \subset \mathbb{H}\setminus K_t$ *with* $\operatorname{diam}(C) < \varepsilon$ *such that* C *separates* $K_{t+\delta}\setminus K_t$ *from infinity in* $\mathbb{H}\setminus K_t$.
- *There is a continuous* $W(t), t \in [0,T]$ *such that* g_t *is the solution of* (3.11).

Proof. The fact that the first statement implies the second one is a straightforward generalization of the proof of Theorem 3.2.1. Namely if $R > 0$ is such that $K_T \subset B(0,R)$ and $t, \varepsilon, \delta, C$ are as in the statement of the theorem, then $\operatorname{diam}(g_t(C)) \leq r_0(\varepsilon) = 6\pi R/\sqrt{\log 1/\varepsilon}$, because $C \subset \overline{B(z_0,\varepsilon)}$ for some $z_0 \in \mathbb{C}$ and by Lemmas 3.2.2 and 3.2.3 there is a circle of radius $\rho \in (\varepsilon, \sqrt{\varepsilon})$ which is mapped by g_t to a curve which has length less than $r_0(\varepsilon)$. Since $g_t(C)$ separates $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta}\setminus K_t)}$ from ∞ in $\mathbb{H}$, also the diameter of $\tilde{K}_{t,\delta}$ is less than $r_0(\varepsilon)$.

The intersection $\bigcap_{s>0} \tilde{K}_{t,s}$ is non-empty because the sets $\tilde{K}_{t,s}$ are compact and any finite intersection is non-empty. Since the diameter of $\bigcap_{s>0} \tilde{K}_{t,s}$ is less than $r_0(\varepsilon')$ for any $\varepsilon' > 0$, there exists $W(t) \in \mathbb{R}$ such that

$$\{W(t)\} = \bigcap_{s>0} \tilde{K}_{t,s}.$$

Now $\tilde{K}_{t,\delta} \subset B(W(t), r_0(\varepsilon))$ and therefore as in (3.10) $t \mapsto W(t)$ is continuous. The Loewner equation now holds by the same argument as in the end of the proof of Theorem 3.2.1. We have shown that the first statement implies the second one.

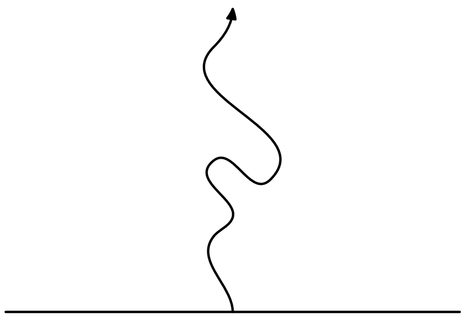
To prove that the second statement implies the first one, define for any $\delta > 0$, the oscillation of W by

$$O(W,\delta) = \sup\{|W(t) - W(s)| \,:\, s,t \in [0,T], |s-t| \leq \delta\}.$$

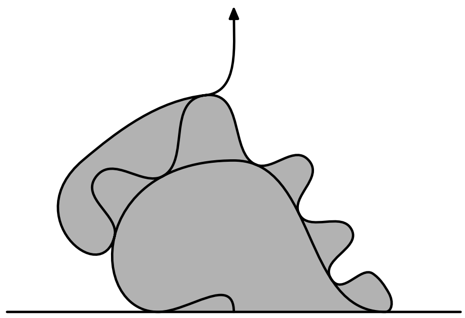
By continuity of W, $O(W,\delta) \searrow 0$ as $\delta \searrow 0$. Let $r_1(\delta) = \left((2\sqrt{\delta})^2 + O(W,\delta)^2\right)^{1/2}$. By the inclusion (3.17), $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta}\setminus K_t)} \subset B(W(t), r_1(\delta))$. By Lemmas 3.2.2 and 3.2.3 there exists an arc of a circle of radius $r \in (r_1(\delta), \sqrt{r_1(\delta)})$

$$S = \mathbb{H} \cap \partial B(W(t), r)$$

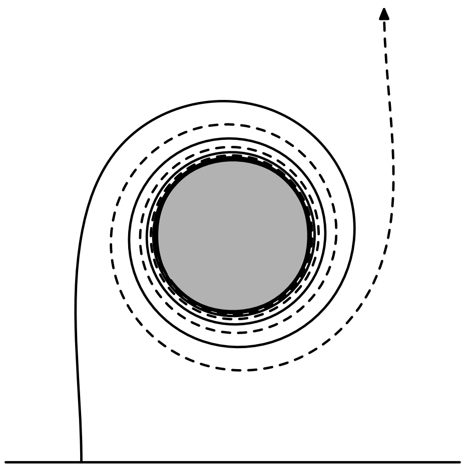
such that the length of $C = g_t^{-1}(S)$ is less than $cR/\sqrt{\log(1/r_1(\delta))}$, where $R > 0$ is such that $K_T \subset B(0,R)$ and $c > 0$ is some universal constant. Since S separates $\tilde{K}_{t,t+\delta}$ from ∞ in $\mathbb{H}$, C separates $K_{t+\delta}\setminus K_t$ from ∞ in H_t. Hence we have existence of the separating set C with a uniformly small diameter. The claim now follows. $\square$



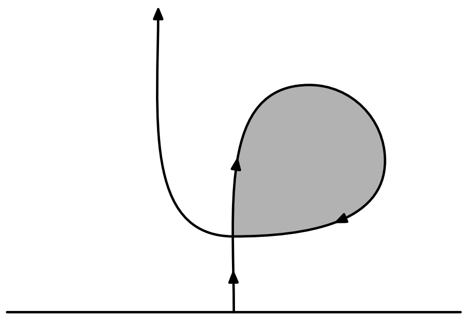
(a) A simple curve in $\mathbb{H}$ starting from the boundary



(b) A non-simple curve which doesn't have self-crossings and the increments $\gamma((t, t+s])$ are always visible to ∞ in $\mathbb{H} \setminus \gamma((0,t])$.



(c) A growing hull, which makes infinite number of turns around a disc



(d) A curve with a self-crossing

Figure 3.3: Some examples and counterexamples based on Theorem 3.2.7: The growing hulls of (a)-(c) satisfy the "local growth" condition but (d) doesn't satisfy the condition. However it is possible to use the Loewner equation for (d), but then the driving term would have a discontinuity at the time of the self-crossing.

Definition 3.2.8. A *Loewner chain* is the solution g_t of the Loewner equation with a continuous driving term.

Remark. By the previous theorem, any one of the quantities $W(t), K_t, g_t$ could be taken as the most fundamental object. Hence the concept of Loewner chain includes all those features.

3.2.3 A historical remark

In 1923, Charles Loewner (his birth name was Karel Löwner in Czech and he used also the name Karl Löwner as a German version of his name) was studying the Bieberbach conjecture in the paper where he introduced the Loewner equation. He was studying conformal maps from the unit-disc, and therefore he introduced the Loewner equation in $\mathbb{D}$ where it is written as

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + e^{iU_t}}{g_t(z) - e^{iU_t}}$$

for a conformal map g_t from a simply connected domain $D_t \subset \mathbb{D}$, $0 \in D_t$, onto $\mathbb{D}$ normalized by the expansion near 0

$$g_t(z) = e^t z + \dots$$

The Loewner equation in $\mathbb{D}$ holds similarly as in Theorem 3.2.7 under some condition of local growth. It holds when $D_t = \mathbb{D} \setminus \gamma((0,t])$ where $\gamma : [0,T] \to \mathbb{C}$ is a simple curve with $\gamma(0) \in \partial\mathbb{D}$ and $\gamma((0,T]) \subset \mathbb{D}$. The function U_t is real and continuous.

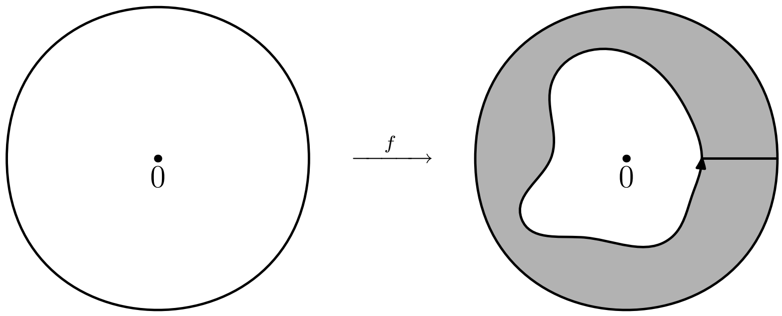


Figure 3.4: A map f from $\mathbb{D}$ into $\mathbb{D}$ can be studied by the Loewner equation in $\mathbb{D}$ by defining a curve that first goes from $\partial\mathbb{D}$ to $\partial f(\mathbb{D})$ and then follows the boundary of the image domain $\partial f(\mathbb{D})$.

Let $0 \in \hat{D} \subset \mathbb{D}$ be a simply connected domain. By approximation we can always assume that the boundary of $\hat{D}$ is a simple curve. By considering a curve $\gamma(t)$, $t \in [0, T]$, as in Figure 3.4 which first follows a curve from $\partial\mathbb{D}$ to $\partial\hat{D}$ (a line segment, say) and then follows $\partial\hat{D}$ in counterclockwise direction, say, we can use the Loewner equation to study the conformal map ϕ from $\hat{D}$ to $\mathbb{D}$ satisfying $\phi(0) = 0$, $\phi'(0) > 0$, because $\phi = g_T$. Using this approach Charles Loewner was able to show that for any $f \in S$ (which has an expansion of the form (2.6))

$$|a_3| \leq 3$$

which is a speacial case of the Bieberbach–de Branges theorem.