Chapter 2

Complex analysis

In this chapter we present briefly some result of complex analysis which are useful for our theory.

A comment to the notation: the complex conjugate $\overline{z}$ ($= z^*$) of a complex number $z = x + i\,y$ is $\overline{z} = x - i\,y$. The upper half-plane and the unit disc are

$$\mathbb{H} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$$
$$\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}.$$

A *domain* is a non-empty, open and connected set.

For a set A, $\overline{A}$ usually denotes the closure, whereas A^* denotes various things often related to the complex conjugation of to other reflections.

2.1 Harmonic functions

Let U be a domain in the complex plane. A twice-continuously differentiable function $u : U \to \mathbb{R}$ is *harmonic* if $\Delta u = 0$. A harmonic function $u : U \to \mathbb{R}$ has mean-value property in the sense that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, \mathrm{d}\theta \tag{2.1}$$

for any $z \in U$ and $r > 0$ such that $\overline{B(0,r)} \subset U$. Conversely, if $u : U \to \mathbb{R}$ is continuous function that has the mean value property (2.1) for every $z \in U$ and for every $0 < r < r_0(z)$ (note that $r_0(z)$ can be strictly less than the distance to the boundary), then u is smooth and harmonic.

When the mean value property is applied together with a Möbius transformation, the mean value property can be written for any point in the disc (not just for the center) as an integral over the boundary of the disc. Namely, if $u : \overline{B(0,R)} \to \mathbb{R}$ is continuous function that is harmonic in $B(0,R)$, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|z - Re^{i\theta}|^2} u(Re^{i\theta})\, \mathrm{d}\theta \tag{2.2}$$

where the quantity

$$P_{B(0,R)}(z,\theta) = \frac{R^2 - |z|^2}{|z - Re^{i\theta}|^2}$$

is called the *Poisson kernel* in $B(0,R)$. This extend to discs $B(z_0,R)$ in obvious way by translation.

Similarly in the upper half-plane, if $u : \overline{\mathbb{H}} \to \mathbb{R}$ is continuous and bounded and if u is harmonic in $\mathbb{H}$ then u is given in terms of an integral of the *Poisson kernel* in $\mathbb{H}$ as

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{|z - \xi|^2} u(\xi)\, \mathrm{d}\xi \tag{2.3}$$

for any $z \in \mathbb{H}$.

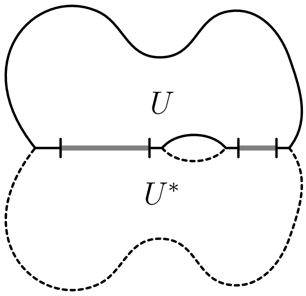


Figure 2.1: By the Schwarz reflection principle, if U is a subdomain of the upper half-plane and a part of the boundary lies on the real axis and if the imaginary part of $f : U \to \mathbb{C}$ vanishes on $J \subset \mathbb{R}$, then f can be extended holomorphically to $U \cup J \cup U^*$

The *harmonic conjugate* of u is any harmonic function v such that $f = u+iv$ is holomorphic. If the function v exists, it is unique up to an additive constant. In a simply connected domains the harmonic conjugate exists. This can be seen from the Poisson kernel which can be written as

$$P_{B(0,R)}(z,\theta) = \operatorname{Re} \frac{Re^{i\theta} - z}{Re^{i\theta} + z}.$$

Therefore if we take the imaginary part of the complex valued kernel $(Re^{i\theta} - z)/(Re^{i\theta} + z)$, then the corresponding integral gives the harmonic conjugate in the disc. This can be summarized by an explicit formula for f in $B(0, R)$ given u

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) \,\mathrm{d}\theta + i\, C$$

where $C \in \mathbb{R}$ is a constant. In simply connected domains, the harmonic extension via overlapping discs is well-defined and hence v exists in the whole domain U. By considering $-if = v - iu$, we see that $-u$ is the harmonic conjugate of v.

Another consequence of the mean value property characterization of harmonic functions is the *Schwarz reflection principle*: if $f = u + iv$ is holomorphic in $D_+ = B(0, r) \cap \mathbb{H}$ and if $\lim v(z) = 0$ as $z \in D_+$ tends to any $x \in (-r, r)$, then f has a unique holomorphic extension to $B(0, r)$. Namely, $v(\overline{z}) = -v(z)$ for any $z \in D_-$ defines a continuous extension of v to $B(0, r)$ and this extension satisfies the mean value property in $B(0, r)$. Hence v is smooth and harmonic in $B(0, r)$ and it has a harmonic conjugate which is unique if we require that $f = u + iv$ is in D_+. Hence f is well-defined and holomorphic in $B(0, r)$ and satisfies

$$f(\overline{z}) = \overline{f(z)}. \tag{2.4}$$

More generally, if $U \subset \mathbb{H}$ is a domain and $J \subset \mathbb{R} \cap \partial U$ is non-empty set such that each point $x \in J$ satisfies the condition that $B(x, r) \cap \mathbb{H} \subset U$ for some $r > 0$ and if $f : U \to \mathbb{C}$ is holomorphic function such that $\lim \operatorname{Im} f(z) = 0$ as z tends to J, then there exists a unique holomorphic extension of f to $U \cup J \cup U^*$ and the extension satisfies (2.4). Here U^* is the reflection of U with respect to the real axis.

Finally, under suitable conditions on the domain U and on the function $u : \overline{U} \to \mathbb{R}$ and its boundary values $\phi = u|_{\partial U}$, a function u which is harmonic in U can be represented using the complex Brownian motion in the following way. Let $\mathbb{P}^z$ be the law of a complex Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ started from z and let $\tau = \inf\{t \in \mathbb{R}_+ : B_t \notin U\}$. Then $\tau < \infty$ almost surely and

$$u(z) = \mathbb{E}^z(\phi(B_\tau))$$

where $\mathbb{E}^z$ denotes the expected value with respect to $\mathbb{P}^z$. This holds, for example when U is a Jordan domain and u is continuous in $\overline{U}$. This is a consequence of the optional stopping theorem, because $u(B_t)$ is a bounded continuous martingale as we saw in the proof of Theorem 1.7.1.

2.2 Conformal maps

Definition 2.2.1. A map $f : U \to \mathbb{C}$ is a *conformal map* if and only if it is holomorphic and injective. A *univalent function* is the same as a conformal map.

Remark. When we consider *conformal and onto maps* $f : U \to U'$, i.e., f is conformal and $f(U) = U'$, we state explicitly that the map is onto.

If we know that a mapping $f : U \to \mathbb{C}$ is holomorphic, then locally near $z_0 \in U$, f is conformal if and only if $f'(z_0) \neq 0$. However, that is not sufficient globally. For example, consider the map $z \to z^2$ in $\mathbb{C} \setminus \{0\}$. Its derivative is non-zero everywhere, but it is not injective because $z^2 = (-z)^2$.

If f is conformal, locally near z_0, we have the absolutely convergent expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots$$

Hence if we ignore the small correction of order $|z - z_0|^2$, locally the map f translates z_0 to $f(z_0)$, rotates around that point by multiplying with the complex number (of unit modulus) $f'(z_0)/|f'(z_0)|$ and scales by the factor $|f'(z_0)|$.

2.2.1 Möbius maps

Definition 2.2.2. A mapping of the form

$$z \mapsto \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc \neq 0$$

is called a *Möbius map* or *fractional linear transformation.*

If $ad - bc = 0$, then either the denominator is identically zero and hence the fraction is not well-defined or the numerator and denominator are constant multiples of each other and hence the fraction is constant. Therefore these maps make sense only when $ad - bc \neq 0$.

Maybe the best way to describe Möbius maps is that they are the conformal self-maps of the *Riemann sphere* (also known as *extended complex plane*) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Exercise. If $A \in \mathbb{C}^{2\times 2}$, $\det A \neq 0$, define a Möbius map by

$$\phi_A(z) = \frac{a_{11}\, z + a_{12}}{a_{21}\, z + a_{22}}$$

where $A_{ij} = a_{ij}$. Show that for any $A, B \in \mathbb{C}^{2\times 2}$

$$\phi_A \circ \phi_B = \phi_{AB}.$$

Use this to find the inverse map of any Möbius map.

Exercise. Show that $\phi : \mathbb{D} \to \mathbb{D}$ is a conformal onto map if and only if

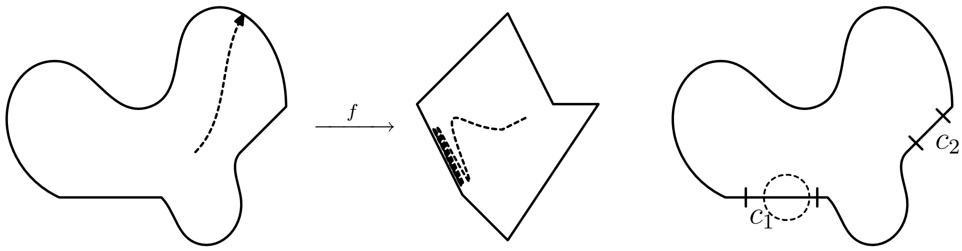
$$\phi(z) = e^{i\theta} \frac{z - a}{1 - \overline{a} z} \tag{2.5}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. *Hint.* For the "only if" claim, use the Schwarz lemma.

Theorem 2.2.3 (Schwarz lemma)**.** *Suppose f is holomorphic on $\mathbb{D}$, $\|f\|_\infty \leq 1$ and $f(0) = 0$. Then*

$$\begin{aligned} |f(z)| &\leq |z| \quad \textit{for any } z \in \mathbb{D}, \\ |f'(0)| &\leq 1. \end{aligned}$$

If equality holds for some z in the first inequality or equality holds in the second inequality, then $f(z) = \lambda z$ for each $z \in \mathbb{D}$, where λ is a constant and $|\lambda| = 1$.



(a) If $f : U \to U'$ is a homeomorphism and $\gamma(t)$ is a curve that tends to the boundary, then the image $f(\gamma(t))$ tends to the boundary. However, it is not always true that $f(\gamma(t))$ extends continuously to its end point, not even when γ extends continuously to its end point.

(b) Schwarz reflection principle can be applied in those boundary arcs that are straight line segments and away from other parts of the boundary.

Figure 2.2: By Theorems 2.2.5 and 2.2.6 conformal map maps boundary to boundary and extends continuously and injectively to a piece of boundary which is a straight line segment, an arc of a circle or an analytic curve.

2.2.2 Riemann mapping theorem

A domain $U \subset \mathbb{C}$ is *simply connected* if its complement $\hat{\mathbb{C}} \setminus U$ in the Riemann sphere is connected. For example $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ is simply connected because the parts $\operatorname{Im} z \leq 0$ and $\operatorname{Im} z \geq 1$ can be connected through infinity. An equivalent definition of simply connectedness is that each closed loop in U is null-homotopic, that is, each loop can be continuously shrunk to a trivial curve. See Rudin's book for more details.

Theorem 2.2.4 (Riemann mapping theorem)**.** *Suppose $U \subset \mathbb{C}$ is a simply connected domain other than $\mathbb{C}$ and $w \in U$. Then there exist a unique conformal map f from U onto $\mathbb{D}$ such that $f(w) = 0$ and $f(w) > 0$.*

Remark. All the other conformal maps from U onto $\mathbb{D}$ are obtained by composing f with a Möbius self-map of the disc which all are of the form (2.5).

2.2.3 Continuity to the boundary

In this section, we follow Ahlfors [1] and Pommerenke [12]. We first state and prove a theorem that shows that any conformal map boundary to the boundary at least in some very weak sense. Then we apply this to the parts of the boundary which are straight line segments. The Schwarz reflection principle will show that there is indeed a continuous extension to those parts of the boundary. Finally we state the general result about continuous extensions of conformal maps to the boundary.

Let's first state and prove a theorem that shows that for any conformal onto map $f : U \to U'$, if a sequence (z_n) or a curve $\gamma(t)$ tends to the boundary of U, then $(f(z_n))$ or $f(\gamma(t))$ tends to the boundary of U'. For that purpose, we have to define what we mean when we say that a sequence or a curve tends to the boundary domain. Let U be a non-empty open set, $z_n \in U$ a sequence and $\gamma : [0, 1) \to U$ a curve. Remember that a *curve* in a topological space X is a continuous map from an interval of $\mathbb{R}$ into X. We say that (z_n) or $\gamma(t)$ *tends to the boundary* if (z_n) or $\gamma(t)$ will stay eventually away from any point in U, more formally, for each $z \in U$ there exist $\varepsilon(z) > 0$ and $n_0(z) \in \mathbb{N}$ such that $|z - z_n| \geq \varepsilon(z)$ for $n \geq n_0(z)$ or there exist $\varepsilon(z) > 0$ and $0 \leq t_0(z) < 1$ such that $|z - \gamma(t)| \geq \varepsilon(z)$ for $t_0(z) \leq t < 1$.

The discs $B(z, \varepsilon(z))$ form an open covering of U and for any compact $K \subset U$ there is a finite subcover. Hence we see that z_n or $\gamma(t)$ will stay eventually away from any compact $K \subset U$ in the sense that there exist $n_0(K) \in \mathbb{N}$ and $0 \leq t_0(K) < 1$ such that $z_n \notin K$ for $n \geq n_0(K)$ and $\gamma(t) \notin K$ for $t_0(K) \leq t < 1$. After noticing this the following theorem is almost trivial.

Theorem 2.2.5. *Let U and U' be non-empty open subsets of $\mathbb{C}$ and let $f : U \to U'$ be a homeomorphism. If (z_n) or $\gamma(t)$ tends to the boundary of U, then $(f(z_n))$ or $f(\gamma(t))$ tends to the boundary of U'.*

Proof. Let $K \subset U'$ be compact. Then $f^{-1}(K)$ is compact and there is $n_0 \in \mathbb{N}$ and $0 \leq t_0 < 1$ such that $z_n \notin f^{-1}(K)$ for $n \geq n_0$ and $\gamma(t) \notin f^{-1}(K)$ for $t_0 \leq t < 1$. Therefore $f(z_n) \notin K$ for $n \geq n_0$ and $f(\gamma(t)) \notin K$ for $t_0 \leq t < 1$. $\square$

Next we state and prove a theorem based on the Schwarz reflection principle that gives the continuity of f to the boundary arcs which are straight line segments.

Suppose that the boundary of U contains an open straight line segment c. By applying rotation and translation, we can assume that c is the interval $a < x < b$ on the real line. It is important that the other parts of the boundary stay away from c. Hence we suppose that every point on c has an open neighborhood in $\mathbb{C}$ whose intersection with the whole boundary ∂U is the same as with just the arc c. By this assumption each point in c is now a center of a disc whose diameter is a subset of c, and which c divides in to two half-discs which are either completely inside or outside of U. Notice that at least one of the half-discs is inside U. Since c is connected, the property, whether one or two half-discs are inside U, is the same in each point. Therefore we can name these cases as *one-sided free arc* and *two-sided free arc*. See Figure 2.2(b) where c_1 and c_2 are one-sided free arcs.

Theorem 2.2.6. *Let U be a domain with one-sided free arc c. Then any conformal onto map $f : U \to \mathbb{D}$ can be extended to a holomorphic and injective map on $U \cup c$. The image of c is an arc c' on the unit circle $\partial\mathbb{D}$. Furthermore, if we apply the same extension to two or more one-sided free arcs, then the resulting extension is holomorphic and injective.*

Proof. Let c be one-sided free arc and $x \in c$ and D a half-disc neighborhood of x which is contained in U. We can assume that the point $f^{-1}(0)$ is not in D by choosing smaller D if necessary. Then $\log f(z)$ has single valued branch in D and its real part tends to 0 as $z \in D$ tends to c, because by the previous theorem $|f(z)|$ goes to 1. Therefore by the Schwarz reflection principle (2.4), $\log f(z)$ has holomorphic extension to $D \cup c \cup D*$ where D^* is the reflection of D with respect to $\mathbb{R}$. Therefore $f(z)$ can be extended holomorphically to a disc around z. The extensions in overlapping disc must coincide and therefore f has holomorphic extension to c and $|f(z)| = 1$ when $z \in c$. Call the neighborhood of c which lies outside U as U_-. Then f is now defined on $U \cup c \cup U_-$.

Clearly the extension is one-to-one if we manage to prove that $f(x) \neq f(x')$ for any $x, x' \in c$, $x \neq x'$ after all in $|f| < 1$ in U, $|f| = 1$ on c and $|f| > 1$ in U_- and in addition in U_-, f is by construction one-to-one. Assume that for some $x, x' \in c$, $x \neq x'$, $f(x) = f(x')$. We can assume $f(x) = 1$. Now $f^{-1}((1-\varepsilon, 1))$ intersects any neighborhood of x and x' which contradicts with the injectivity of f in U. A similar argument gives the last claim. $\square$

Remark. The previous theorem has an extended version to the case when c is an arc of a circle or even more generally when c an image of line segment under a homolomophic map (c is said to be an *analytic arc*).

A compact set $A \subset \mathbb{C}$ is said to be *locally connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any two points $a, b \in A$ with $|a - b| < \delta$, there exist a closed connected set B with $a, b \in B \subset A$ and $\operatorname{diam} B < \varepsilon$. For non-bounded closed $A \subset \mathbb{C} \cup \{\infty\}$, we could adjust this definition and the next theorem by defining metric on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ that makes $\mathbb{C} \cup \{\infty\}$ a compact space (See Problem sheet 6, Exercise 5).

Theorem 2.2.7. *Let $U \subset \mathbb{C}$ be a bounded domain. A conformal onto map $f : \mathbb{D} \to U$ extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ if and only if ∂U is locally connected.*

If $f : \mathbb{D} \to U$ is as in the previous theorem and if it extends continuously to the boundary, then obviously ∂U is a closed curve that can be parametrized as $\theta \mapsto f(e^{i\theta})$. On the other hand, any closed curve is locally connected. Hence f extends continuously to the boundary if and only if the boundary is a curve. Clearly this extension is injective if and only if $\theta \mapsto f(e^{i\theta})$ is a simple curve. Hence the previous theorem implies that f extends to a continuous and injective map from $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if U is a Jordan domain. In fact, the inverse map is in that case continuous to the boundary and any conformal map between two Jordan domains extends to a homeomorphism between their closures.

2.3 From Area theorem to distortion

In this section we present some classical result about the following two classes of functions:

- The class S consists of all functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad |z| < 1 \tag{2.6}$$

holomorphic and univalent in $\mathbb{D}$.

- The class Σ consists of all functions

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \quad |z| > 1 \tag{2.7}$$

holomorphic and univalent in $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$.

Note that if $f \in S$, then

$$g(z) = 1/f(z^{-1}) = z - a_2 + (a_2^2 - a_3) z^{-1} + \dots \tag{2.8}$$

belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$. Conversely if g belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$, then

$$f(z) = 1/g(z^{-1}) = z - b_0 z^2 + (b_0^2 - b_1) z^3 + \dots$$

belongs to S.

If $g \in \Sigma$ and we apply Stokes' theorem to calculate the area inside $\theta \mapsto g(re^{i\theta})$, $r > 1$, we get the following formula, for the proof see [3], p. 29:

$$\text{area}\,(\mathbb{C} \setminus g\,(\mathbb{D}^*)) = \pi \left(1 - \sum_{n=1}^{\infty} n |b_n|^2 \right)$$

from which the next theorem follows.

Theorem 2.3.1 (Area theorem)**.** *For any $g \in \Sigma$,*

$$\sum_{n=1}^{\infty} n |b_n|^2 \leq 1$$

If $f \in S$ and the coefficients are as in (2.6), then there exists odd function $h \in S$ such that $h(z) = \sqrt{f(z^2)}$ and

$$h(z) = z + \frac{1}{2} a_2 z^3 + \dots$$

The function h can be constructed as follows: The function $\phi(z) = \log(f(z)/z)$ has single-valued branch in $\mathbb{D}$, because $f(z)/z$ is holomorphic and doesn't have zeros in $\mathbb{D}$. Choose the branch for instance so that $\phi(0) = 0$. Hence $f(z) = z \exp \phi(z)$ and $h(z) = z \exp(\phi(z^2)/2)$ is in S and satisfies the required properties. Therefore (2.8) and the Area theorem imply that for any $f \in S$

$$|a_2| \leq 2. \tag{2.9}$$

This is a special case of the following famous and difficult theorem. As a historical remark, Charles Loewner invented the Loewner equation to study one of the special cases of the Bieberbach conjecture.

Theorem 2.3.2 (Bieberbach conjecture – de Branges theorem)**.** *For any $f \in S$, $|a_n| \leq n$, $n = 2, 3, \dots$*

To apply (2.9) to more general setting, define for any f univalent in $\mathbb{D}$ and for any $w \in \mathbb{D}$ a function

$$h(z) = \frac{f\left(\frac{z+w}{1+\overline{w}\,z}\right) - f(w)}{(1 - |w|^2)\, f'(w)} = z + \left(\frac{1}{2}(1 - |w|^2) \frac{f''(w)}{f'(w)} - \overline{w} \right) z^2 + \dots$$

We leave as an exercise to verify the expansion. Since $h \in S$, this expansion and (2.9) imply the following result.

Proposition 2.3.3. *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ and if $z \in \mathbb{D}$ then*

$$\left| (1 - |z|^2) \frac{f''(z)}{f'(z)} - 2\overline{z} \right| \leq 4.$$

The previous result can be integrated (see [3]), p. 32) to give Koebe's theorem which tells, for example, how f distorts the circle $\theta \mapsto re^{i\theta}$. The first of the inequalities tells that $\theta \mapsto f(re^{i\theta})$ lies between two particular circles centered at $f(0)$ and the second inequality tells that the length of this curve is bounded from below and from above by certain constants.

Theorem 2.3.4 (Koebe distortion theorem)**.** *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ and if $z \in \mathbb{D}$ then*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z)-f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}$$
$$|f'(0)|\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)|\frac{1+|z|}{(1-|z|)^3}$$

One of the consequences of the following. Here $\mathrm{dist}(x, A)$ is the Euclidian distance from a point x to a set A.

Theorem 2.3.5. *Let $f \in S$ and $U = f(\mathbb{D})$ then*

$$\frac{1}{4} \leq \mathrm{dist}(0, \partial U) \leq 1$$

Proof. Let $f \in S$. Then $f(0) = 0$ and $f'(0) = 1$. Therefore by the Koebe distortion theorem $|f(re^{i\theta})| \geq r/(1+r)^2$. Hence ∂U lies outside $\overline{B(0, r/(1+r)^2)}$ for any $0 < r < 1$. The limit $r \nearrow 1$ gives the inequality on the left.

Let $d = \mathrm{dist}(0, \partial U)$. Define a conformal map g from $\mathbb{D}$ into $\mathbb{D}$ by $g(z) = f^{-1}(d\,z)$. Now $g'(0) = d/f'(0) = d$ and by the Schwarz lemma (Theorem 2.2.3) $|g'(0)| \leq 1$. $\square$