Energy momentum tensor

In general relativity $G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$ (Einstein's field equations) where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, $R^{\mu\nu}$ is the Ricci tensor and $R = g_{\mu\nu} R^{\mu\nu}$ is the Ricci scalar. G is Newton's gravitational constant and c = velocity of light. The field equations are obtained by a variation of the Einstein-Hilbert action with respect to the metric $g_{\mu\nu}$ ($\leftarrow$ left-hand-side). The source term on the right is the energy-momentum tensor due to various matter fields. Thus in a complete model with metric and matter fields

$$T^{\mu\nu} \propto \frac{\delta L_{matter}}{\delta g_{\mu\nu}}$$

where L_{matter} is the Lagrange density of matter fields. Let us apply this to the case of WZW Lagrangian. The term $\Gamma(g)$ is topological and does not depend on the metric $g_{\mu\nu}$, so

$$T^{\mu\nu} \propto \frac{\delta L_{WZW}}{\delta g_{\mu\nu}} = \frac{\delta}{\delta g_{\mu\nu}} \operatorname{tr} \partial_\mu g \, \partial^\mu g^{-1} \cdot \sqrt{-\det(g)}$$

where we have added the factor $\sqrt{-\det(g)}$ in order that the integration measure

$$dx^0 \, dx^1 \cdot \sqrt{-\det(g)}$$

is reparametrization invariant. Clearly $T^{\mu\nu} = T^{\nu\mu}$ but in a conformally invariant action we also have the trace identity

$$T_\mu{}^\mu = g_{\mu\nu} T^{\mu\nu} = 0 .$$

For this reason the energy-momentum tensor of the WZW model has only <u>two independent components</u>.

In the case of the Minkowski metric $(g_{\mu\nu}) = \operatorname{diag}(+1, -1)$, using the light cone coordinates $x_\pm$, we can choose as the independent components

But the product $(\alpha, \beta) \longmapsto \alpha\beta' - \beta\alpha'$ is just the commutator of vector fields

$$\left[\alpha \frac{d}{dx}, \beta \frac{d}{dx}\right] = (\alpha\beta' - \beta\alpha') \frac{d}{dx}.$$

Thus the l_α's are interpreted as generators of the Lie algebra of $\mathrm{Diff}(S^1)$. Setting $l_n := -i\, l_\alpha$ for $\alpha = e^{-inx}$, $n \in \mathbb{Z}$, we get

$$\{l_n, l_m\} = (n-m)\, l_{n+m},$$

the Witt algebra.

$$\{l_\alpha, h_c\} = X_\alpha \cdot h_c = -\int \mathrm{tr}\, c(x)\, (\alpha\xi)'(x)\, dx$$

$$= \int \mathrm{tr}\, \alpha\, \dot{c}(x)\, \xi(x)\, dx = h_{\alpha\dot{c}},$$

so l_α generates diffeomorphisms acting on the parameter $c(x)$. In particular, l_0 corresponding to l_α with $\alpha(x) \equiv 1$ generates rotations on the circle S^1.

Choose a normalized basis $\{T_1, \dots, T_N\}$ in $\underline{g}$,

$$\langle T_i, T_j \rangle := \mathrm{tr}\, T_i T_j = -\delta_{ij}$$

and define $h^n_a(\xi) := \int e^{-inx}\, \mathrm{tr}\, T_a\, \xi(x)\, dx$.

Then

$$\{h^n_a, h^m_b\} = \lambda^c_{ab}\, h^{n+m}_c + ink\, \delta_{n+m}\, \delta_{ab}$$

where $[T_a, T_b] = \lambda^c_{ab}\, T_c$. Furthermore,

$$\{l_n, h^m_a\} = -m\, h^{n+m}_a.$$

Since h^n_a give the Fourier modes of the function $\xi(x)$, we also have

$$l_n = \frac{-i}{2k} \sum_{p,a} h^a_p\, h^a_{n-p}.$$

This is the classical (non quantized) form of the <u>Sugawara formula</u>.

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$$T_{++} \propto \mathrm{tr}\, \partial_+ g\, \partial_+ g^{-1}, \quad T_{--} \propto \mathrm{tr}\, \partial_- g\, \partial_- g^{-1}.$$

Using the factorization $g = A(x_-)B(x_+)$ for the critical value $\lambda^2 = 2\pi/n$ we note that

$$T_{--} \propto \mathrm{tr}\, \partial_- g\, \partial_- g^{-1} = -\mathrm{tr}\,(\partial_- g g^{-1})^2$$

$$= -\mathrm{tr}\,(\partial_- A \cdot A^{-1})^2 \equiv -\mathrm{tr}\,\xi^2.$$

Thus the Fourier components of T_{--} define the Virasoro functions l_m. Similarly, the Fourier components of T_{++} define a commuting Virasoro algebra $\bar{l}_m$.

Quantization using representation theory

Basic observables: 1) Components of the energy momentum tensor, $T_{--} \rightsquigarrow \{l_m\}$, 2) the currents $h_c \rightsquigarrow$ Fourier components $\{h_a^m\}$ with Poisson brackets

$$\{l_n, l_m\} = (n-m) l_{n+m}, \quad \{l_m, h_a^n\} = -m\, h_a^{m+n}$$

$$\{h_a^n, h_b^m\} = \lambda_{ab}^c\, h_c^{n+m} + i n k\, \delta_{n+m}\, \delta_{a,b}$$

where $[T_a, T_b] = \lambda_{ab}^c T_c$ is a basis of $\underline{g}$ with

$$\mathrm{tr}\, T_a T_b = -\delta_{a,b}.$$

In quantum theory we want to construct linear operators in a (dense domain of) Hilbert space with similar commutation relations. The integral of the component T_{--} is interpreted as the energy of the system; it is represented as the element L_0 in the Virasoro algebra and the physical requirement is positivity, $L_0 \geq 0$.

We denote the linear operators corresponding to h_a^n by J_a^n,

$$[J_a^n, J_b^m] = \lambda_{ab}^c J_c^{n+m} + nk\,\delta_{n+m}\,\delta_{a,b}$$

In this basis the unitarity relations are $L_n^* = L_{-n}$, $(J_a^n)^* = -J_a^{-n}$ (note the sign!). Finally,

$$[L_n, J_a^m] = -m J_a^{m+n}.$$

Since $[L_0, J_a^m] = -m J_a^m$, the requirement that there is a vacuum vector v_0, $L_0 v_0 = h v_0$, and $L_0 \geq h$, leads to

$$J_a^m v_0 = 0 \text{ for } m>0.$$

Define

$$:J_a^n J_b^m: = \begin{cases} J_b^m J_a^n & \text{when } a=b,\ n=-m>0 \\ J_a^n J_b^m & \text{otherwise} \end{cases}$$

<u>Theorem</u> (Quantum Sugawara) The operators

$$L_n = \frac{1}{+Q+2k} \sum_{a,k} :J_a^{n+k} J_a^{-k}:$$

satisfy the commutation relations of the Virasoro algebra with the central charge $z = 2k \dim \mathfrak{g} / (-Q+2k)$ where

$$\delta_{bc} \cdot Q = \sum_{a,f} \lambda_{ab}^f \lambda_{af}^c \cdot \cancel{8},$$

and Q is the eigenvalue of the Casimir operator $\sum_a J_a J_a$ in the adjoint representation.

<u>Proof.</u> We first compute

$$\sum_{j,a} [:J_a^{j+n} J_a^{-j}:, J_b^m]$$

$$= \sum J_a^{j+n} [J_a^{-j}, J_b^m] + \sum [J_a^{j+n}, J_b^m] J_a^{-j}$$

$$= \sum \overline{T}_a^{n+\gamma}\left(\lambda_{ab}^c \overline{T}_c^{m-\gamma} + k\,\delta_{a,b}\,\delta_{-\gamma+m}\cdot(-\gamma)\right)$$

$$+ \sum \left(\lambda_{ab}^c \overline{T}_c^{m+n+\gamma} + k\,\delta_{a,b}\,\delta_{\gamma+n+m}\cdot(\gamma+n)\right)\overline{T}_a^{-\gamma}.$$

However, the infinite sums separately are ill-defined; we must normal order:

$$\cdots = \cancel{\sum_{\gamma<m} \lambda_{ab}^c \overline{T}_a^{n+\gamma}\overline{T}_c^{m-\gamma}} + \cancel{\sum_{\gamma\ge m} \lambda_{ab}^c \overline{T}_c^{m-\gamma}\overline{T}_a^{n+\gamma}}$$

$$+ \sum_{\gamma\ge m} \lambda_{ab}^c \lambda_{ac}^f \overline{T}_f^{n+m}$$

$$+ \cancel{\sum_{\gamma<0} \lambda_{ab}^c \overline{T}_c^{m+n+\gamma}\overline{T}_a^{-\gamma}} + \cancel{\sum_{\gamma\ge 0} \lambda_{ab}^c \overline{T}_a^{-\gamma}\overline{T}_c^{m+n+\gamma}}$$

$$+ \sum_{\gamma\ge 0} \lambda_{ab}^c \lambda_{ca}^f \overline{T}_f^{m+n} - 2km\,\overline{T}_b^{n+m}$$

$$= \sum_{0\le\gamma<m} \lambda_{ab}^c \lambda_{ca}^f \overline{T}_f^{m+n} - 2km\,\overline{T}_b^{n+m}$$

$$= -m\cdot Q\cdot \overline{T}_b^{m+n} - 2km\,\overline{T}_b^{n+m} = -m\cdot(+Q+2k)\,\overline{T}_b^{n+m}.$$

Thus $[L_n, \overline{T}_b^m] = -m\,\overline{T}_b^{m+n}$.

Now $[L_n, L_m] = \left[L_n, \frac{1}{-Q+2k}\sum :\overline{T}_a^{m+k}\overline{T}_a^{-k}:\right]$

$$= \frac{1}{+Q+2k}\sum [L_n, \overline{T}_a^{m+\gamma}]\,\overline{T}_a^{-\gamma}$$

$$+ \frac{1}{+Q+2k}\sum \overline{T}_a^{m+\gamma}[L_n, \overline{T}_a^{-\gamma}]$$

$$= \frac{1}{+Q+2k}\sum(-m-\gamma)\,\overline{T}_a^{n+m+\gamma}\,\overline{T}_a^{-\gamma}$$

$$+ \frac{1}{+Q+2k}\sum \overline{T}_a^{m+\gamma}\cdot\gamma\cdot\overline{T}_a^{n-\gamma}$$

$$= (n-m)\frac{1}{+Q+2k}\sum :J_a^{n+m+j}J_a^{-j}:$$

$$+\ \delta_{n+m}\cdot\frac{1}{+Q+2k}\dim\mathfrak{g}\cdot k\Big(\sum_{j>0}(n-j)j+\sum_{j>n}j(-n+j)\Big)$$

$$= (n-m)L_{n+m}+\delta_{n+m}\cdot\frac{\dim\mathfrak{g}}{-Q+2k}\,k\sum_{0<j\le n}j(n-j)$$

$$= (n-m)L_{n+m}+\delta_{n+m}\cdot\frac{\dim\mathfrak{g}}{-Q+2k}\cdot k\cdot\frac{1}{6}n(n^2-1)$$

$$= (n-m)L_{n+m}+\delta_{n+m}\cdot\frac{z}{12}\,n(n^2-1).\quad\square$$

Note that with our conventions in an unitary highest weight representation $k\le 0$:

$$J_a^n v_0 = 0 \text{ for } n>0:\ \langle J_a^{-n}v_0, J_a^{-n}v_0\rangle$$

$$= \langle v_0, -J_a^{n}J_a^{-n}v_0\rangle = -\langle v_0,[J_a^n,J_a^{-n}]v_0\rangle$$

$$= -kn \ge 0 \implies \underline{k\le 0}.$$

Thus $z = 2k\dfrac{\dim\mathfrak{g}}{+Q+2k}\ge 0$.

[With our normalizations, for $G=SU(n)$, $Q=-2n$.]

The basic representation, $G = SU(n)$

Start from complex fermion algebra

$$[b_i, b_j]_+ = [b_i^*, b_j^*]_+ = 0,\quad [b_i, b_j^*]_+ = \delta_{ij}.$$

If $X = (X_{ij})$ is an infinite matrix we set

$$\hat{X} = \sum_{i,j} X_{ij} :b_i^* b_j:$$

where

$$:b_i^* b_j: = \begin{cases} -b_j b_i^* & \text{for } i=j>0 \\ b_i^* b_j & \text{otherwise}\end{cases}$$

The operators are acting in a fermionic Fock space with vacuum v_0,

$$b_n^* v_0 = 0 = b_{-m} v_0 \quad \text{for } n \geq 0.$$

We have to assume that $\sum_{i\geq 0 > j} |X_{ij}|^2 < \infty$ and $\sum_{i<0\leq j} |X_{ij}|^2 < \infty$. We also assume that (X_{ij}) defines a bounded operator in the Hilbert space ℓ_2 of sequences $a = (a_n)_{n\in\mathbb{Z}}$, $\sum |a_n|^2 < \infty$.

Theorem $[\hat{X}, \hat{Y}] = \widehat{[X,Y]} + c(X,Y)$, where $[X,Y]$ is the ordinary matrix commutator and

$$c(X,Y) = -\operatorname{tr}\,(X_{+-}Y_{-+} - X_{-+}Y_{+-})$$
$$= -\tfrac{1}{2}\operatorname{tr}\,[\varepsilon, X]Y$$

where $X = \begin{pmatrix} X_{++} & X_{+-} \\ X_{-+} & X_{--} \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

with respect to the polarization to the subspace $n \geq 0$ and $n < 0$ for the indices i,j in (X_{ij}).

Proof By the bilinearity of the commutators it is sufficient to test the relations for the elementary operators $\hat{e}_{ij} = :b_i^* b_j:$.

Now $[\hat{e}_{ij}, \hat{e}_{kl}] = [b_i^* b_j, b_k^* b_l] =$

$[b_i^* b_j, b_k^*] b_l + b_k^* [b_i^* b_j, b_l] =$

$b_i^* [b_j, b_k^*]_+ b_l - b_k^* [b_i^*, b_l]_+ b_j$

$= \delta_{jk} b_i^* b_l - \delta_{il} b_k^* b_j$

$= \delta_{jk} :b_i^* b_l: - \delta_{jk}\delta_{il}\,\Theta(i) - \delta_{il} :b_k^* b_j: + \delta_{il}\delta_{kj} \times \Theta(k)$

where $\theta(i)=1$ for $i>0$ and $\theta(i)=0$ for $i<0$. But the right-hand-side is equal to
$$\delta_{jk}\,\hat{e}_{il} - \delta_{il}\,\hat{e}_{kj} + c(e_{ij}, e_{kl}). \quad \Box$$

Note that the Hilbert-Schmidt conditions for the off-diagonal blocks of X, Y are needed in order that $c(X,Y)$ is well-defined, absolutely converging, for infinite matrices.

<u>Exercise</u> Show that $X v_0$ has a finite norm in the Fock space when X_{+-}, X_{-+} are Hilbert-Schmidt.

<u>Example</u> Let $\{v_n\}_{n\in\mathbb{Z}}$ be the standard basis of l_2 and define
$$S_k : l_2 \to l_2, \quad S_k v_n := v_{n+k}$$
for any $k\in\mathbb{Z}$. Then
$$c(S_k, S_l) = -\tfrac{1}{2}\,\mathrm{tr}\,[\varepsilon, S_k]S_l = -k\cdot\delta_{k+l}.$$
Using the Fourier basis $v_n = \frac{1}{\sqrt{2\pi}}e^{inx}$ on S^1 we can identify $L^2(S^1)\cong l_2$ and the operator S_k is the same as multiplication by the Fourier mode $\exp(ikx)$.

Next let $H = L^2(S^1, \mathbb{C}^N)$ and define the polarization ε as before by splitting H to positive Fourier modes and nonpositive modes (each mode now with multiplicity N). Let again $\{T_a\}$ be a basis of $\underline{g} = \underline{su}(n)$ with
$$\mathrm{tr}(T_a T_b) = -\delta_{ab},$$
$$T_a^n = e^{inx}\,T_a. \text{ Then } [T_a^n, T_b^m] = \lambda_{ab}^c\, T_c^{n+m}$$
but $c(T_a^n, T_b^m) = -n\,\delta_{n+m}\,\delta_{ab}$, so
$$[\hat{T}_a^n, \hat{T}_b^m] = \lambda_{ab}^c\,\hat{T}_c^{n+m} - n\,\delta_{n+m}\,\delta_{ab}.$$

There is a complete classification of unitary highest weight representations of the central extension $\hat{L\underline{g}} = L\underline{g} \oplus \mathbb{C}$ of the loop algebra $L\underline{g}$, defined by

$$[J_a^n, J_b^m] = \lambda_{ab}^c J_c^{m+n} + nk\,\delta_{m+n}\cdot\delta_{a,b}.$$

The representations are parametrized by a <u>highest weight</u> $\lambda \in \underline{h}^*$ of the Lie algebra $\underline{g}$ and the <u>level</u> k. Here $\underline{h} \subset \underline{g}$ is a <u>Cartan subalgebra</u>; for example, for $\underline{g} = su(n)$ we may take $\underline{h}$ = diagonal matrices in $\underline{g}$. A basis for $\underline{h}$ is given by the matrices

$$h_i = e_{ii} - e_{nn} \quad (\text{in } \underline{g}_{\mathbb{C}}), \quad i = 1, 2, \dots, n-1.$$

A highest weight representation is characterized by the existence of a cyclic vector v_0 s.t.

1) $h_i v_0 = \lambda_i v_0 \qquad i = 1, 2, \dots, n-1$

2) $J_a^n v_0 = 0$ for $n > 0$

3) $e_{ij} v_0 = 0$ for $i < j$.

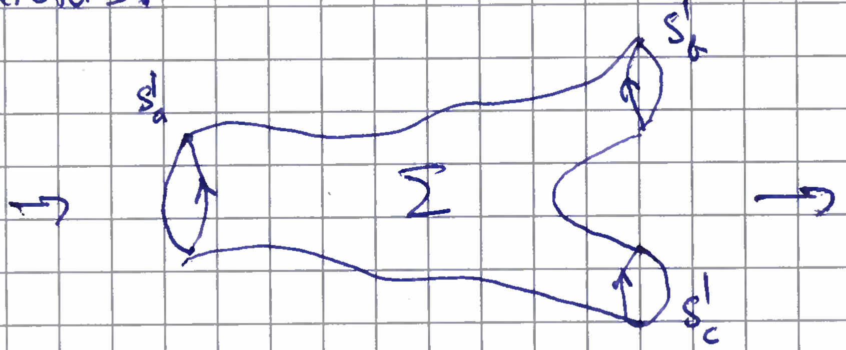
Here we use $\{e_{ij}\}$ as a basis in the 0-Fourier mode sector $n = 0$. In an unitary highest weight representation (for $\underline{g} = su(n)$) one has

$$\lambda_i = 0, 1, 2, \dots; \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1}; \quad \lambda_1 \leq -k; \quad -k = 0, 1, 2, \dots$$

Conformal field theory on Riemann surfaces (Segal's axioms)

Graeme Segal's axiom system is motivated by applications to string theory and is closely related to M. F. Atiyah's axioms for a topological field theory.

In short, a conformal field theory is defined as a functor from the category of closed 1-dimensional manifolds (collections of circles) and their morphisms defined as Riemann surfaces with the given boundary circles, to the category of (infinite dimensional) vector spaces and morphisms given by Hilbert-Schmidt operators.



$\mathcal{C}$ = the category of collections of parametrized circles S^1

morphisms : Oriented Riemann surfaces Σ s.t. $\partial\Sigma$ = the given collection of circles, compatible with orientations

$\mathcal{V}$ = category of vector spaces of type $H \otimes H \otimes \dots \otimes H \equiv H^{\otimes n}$, one factor for each 'outgoing' circle; and vector spaces $\bar{H} \otimes \bar{H} \otimes \dots \otimes \bar{H} \equiv \bar{H}^{\otimes m}$, one factor for each 'incoming' circle.

morphism : linear map $H^{\otimes m} \longrightarrow H^{\otimes n}$ or equivalently a vector in

$$\bar{H}^{\otimes m} \otimes H^{\otimes n}.$$

In string theory one deals also with indefinite metric spaces, but we shall here assume that H is a Hilbert space and $\overline{H}$ its dual. In that case a vector in $\overline{H}^{\otimes m} \otimes H^{\otimes n}$ is the same as a Hilbert-Schmidt map $H^{\otimes m} \to H^{\otimes n}$.

Recall: Let E, F be Hilbert spaces and $\{e_i\}$ a basis of E and $\{f_i\}$ a basis of F then a H-S operator $A: E \to F$ is in matrix form

$$A e_i = \sum_j A_{ji} f_j \quad \text{with} \quad \sum_{i,j} |A_{ij}|^2 < \infty.$$

But this can be viewed as a vector in $\overline{E} \otimes F$,

$$v = \sum A_{ji} \overline{e}_i \otimes f_j$$

where $\{\overline{e}_i\}$ is the dual basis, $\langle \overline{e}_i, e_j \rangle = \delta_{ij}$. This correspondence does not depend on basis: Any vector

$$v = \sum x_i \otimes y_i \qquad x_i \in \overline{E}, \quad y_i \in F$$

defines a map $E \to F$ by $u \mapsto \sum_i \langle x_i, u \rangle y_i$.

In conformal field theory one requires that the functor $C \to V$ is reparametrization invariant: A diffeomorphism of the surface Σ should not change the vector $\mathfrak{F}(\Sigma) \in \overline{H}^{\otimes m} \otimes H^{n}$, when the diffeomorphism is the identity on the boundary $\partial\Sigma$.

Conformal anomaly: There is a slight modification. The vector $\mathfrak{F}(\Sigma)$ might be defined only up to a projective phase factor $e^{i\alpha} \mathfrak{F}(\Sigma)$.

In addition, one has the following axioms:

c1) The permutation of the boundary components S^1_a (in), $a = 1, 2, \dots m$ or S^1_b (out), $b = 1, 2, \dots, n$ corresponds to a permutation of the coordinates in $\overline{H}^{\otimes m}$ or in $H^{\otimes n}$

c2) "Crossing symmetry": Reversing the orientation of the incoming boundary components associates to $\mathcal{F}(\Sigma)$ a new vector in $H^{\otimes m} \otimes H^{\otimes n}$ (only outgoing boundary components)

c3) Sewing property: Glue $\Sigma_1 \cup \Sigma_2$ along the outgoing boundary of Σ_1 and the incoming boundary of Σ_2 (same number of components!). This gives a linear operator $H^{\otimes m_1} \longrightarrow H^{\otimes n_2}$ which is a composition $\mathcal{F}(\Sigma_2) \circ \mathcal{F}(\Sigma_1)$. In particular, when $m_1 = n_2 = 0$ the surface $\Sigma_1 \cup \Sigma_2$ is closed and $\mathcal{F}(\Sigma_1 \cup \Sigma_2) \in \mathbb{C}$.

c4) "Reflection symmetry": $\overline{\Sigma}$ complex conjugate surface of Σ (opposite orientation), then
$$\mathcal{F}(\overline{\Sigma}) = \mathcal{F}(\Sigma)^*, \text{ adjoint operator.}$$

Example: Free chiral fermions

Consider 1-component (left-handed) fermions $\psi(z)$ on a Riemann surface. It is convenient to think ψ as a "half form" $\psi = \psi(z)(dz)^{\frac{1}{2}}$ since this encodes the spinorial structure of ψ in coordinate transformations:
$$\psi'(z') = \left(\frac{dz}{dz'}\right)^{\frac{1}{2}} \psi(z).$$

Any 2-dim. Riemann surface has a spin structure:

The signs of the square roots can be chosen in a consistent way.

if the surface is just the unit disc D, then the restriction of holomorphic $\frac{1}{2}$-forms to the boundary circle S^1 consists of smooth $\frac{1}{2}$-forms $f(\theta)\,(d\theta)^{1/2}$ on the boundary with only nonnegative Fourier components $f(\theta) = \exp(in\theta)$. This leads to our definition of

$$H = H_+ \oplus H_- = \{ f \in L^2(S^1) \}$$

with a polarization to nonnegative and negative Fourier modes. The holomorphicity condition

$$\bar{\partial}\psi = 0, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}, \quad z = x + iy$$

is actually the Dirac equation for 1-component complex fermions.

Let $\mathcal{H}$ be the fermionic Fock space with vacuum $v_0 \in \mathcal{H}$, $\|v_0\| = 1$,

$$b_n v_0 = 0 \text{ for } n \geq 0, \quad b_n^* v_0 = 0 \text{ for } n < 0.$$

Here $[b_n, b_m]_+ = [b_n^*, b_m^*]_+ = 0$ and

$$[b_n^*, b_m]_+ = \delta_{n-m}.$$

In the Fock space b_m^* is the adjoint of b_m and the space $\mathcal{H}$ is a completion of the space of polynomials in $\{b_m, b_m^*\}$ acting on v_0.

Our functor $\mathcal{F}$ is now defined: Associate to each circle (parametrized) S^1 the Hilbert space $\mathcal{H}$ and to each Riemann surface Σ the following vector in $\mathcal{H}^{\otimes m} \otimes \mathcal{H}^{\otimes n}$ (m incoming, n outgoing boundary circles): Let $W \subset H^{\oplus m} \oplus H^{\oplus n}$ be the closure of the subspace obtained by restricting holomorphic fields ψ to the boundary $\partial\Sigma$.

<u>Proposition</u> Let $H^{n,m} = H_+^{n,m} \oplus H_-^{n,m}$ be the polarization of the boundary space $H^{n,m} = \bar{H}^{\oplus m} \oplus H^{\oplus n}$ to nonnegative and negative Fourier modes. Then the orthogonal projection $\pi_- : W \to H_-^{n,m}$ is Hilbert-Schmidt, the projection $\pi_+ : W \to H_+^{n,m}$ is Fredholm (this latter property follows from the former).

<u>Proof:</u> See A. Pressley, G. Segal: Loop Groups, Prop. 8.11.10.

Any closed subspace $W \subset H^{n,m}$ with the above properties define a ray in the Fock space $\mathcal{H}^{n,m}$. There is a normalized vector Ψ_W, uniquely defined up to a phase $e^{i\alpha}\Psi_W$, such that

$$\begin{cases} b(w)\Psi_W = 0 & \forall\, w \in W \\ b^*(w)\Psi_W = 0 & \forall\, w \in W^\perp. \end{cases}$$

Here $b^*(w) := \sum_n w_n b_n^*$ for $w = \sum w_n e_n$ where $\{e_n\}$ is a basis of the boundary subspace (indexing compatible with the polarization) and $b(w) = \sum_n \bar{w}_n b_n$.

We define the functor $\mathcal{F}(\Sigma) := \Psi_W \in \mathcal{H}^{n,m}$. This is really only a projective functor, the phase of Ψ_W is not fixed.

We skip the proof of the properties (1) – (4), but for example the sewing property (3) follows essentially from the fact that holomorphic fields on Σ_1, Σ_2 can be glued together on the common boundary to holomorphic fields on $\Sigma_1 \cup \Sigma_2$.

REF.: G. Segal: The definition of conformal field theory London Math. Soc. Lecture Notes Series 308 (2004)