# IV WESS - ZUMINO - WITTEN MODEL

Consider functions $g = g(x^0, x^1)$ in $\mathbb{R}^{1,1}$ taking values in a compact simply connected Lie group $G$; for example, $G = SU(n) = \{ g \in U(n) \mid \det(g) = 1 \}$.

Fix a finite-dimensional unitary representation $\rho$ of $G$. We shall use the short notation $\rho(g(x)) = g(x)$ in the following. Define the Lagrange function

$$L(g) := \frac{1}{4\lambda^2} \int d^2x \ \mathrm{tr}\, \partial_\mu g \, \partial^\mu (g^{-1}) + n\Gamma(g)$$

where $\lambda > 0$ is a constant, $n \in \mathbb{Z}$, and

$$\Gamma(g) = \frac{1}{12\pi} \int_B d^3x \ \varepsilon^{ijk} \ \mathrm{tr} (\tilde{g}^{-1} \partial_i \tilde{g})(\tilde{g}^{-1} \partial_j \tilde{g})(\tilde{g}^{-1} \partial_k \tilde{g})$$

and $\tilde{g}$ is a smooth extension of $g$ to three dimensions: First, assume that $g$ is constant at infinity, $|x| \to \infty$. Then we may think of $g$ as a function on $S^2$ (= the 1-point compactification of $\mathbb{R}^2$). It is a topological fact that any smoothe function $S^2 \to G$ extends to a function $\tilde{g}: B \to G$, where $B \subset \mathbb{R}^3$ is the closed unit disk with boundary $S^2 = \partial B$ [that is $\pi_2(G) = 0$ for any finite-dimensional Lie group $G$].

The value of $\Gamma(g)$ does depend on the extension $\tilde{g}$, but only up to $2\pi \times$ integer: The difference $\Gamma(\tilde{g}) - \Gamma(g')$ for two extensions $\tilde{g}, g'$ can be written as an integral

$$\Gamma(\tilde{g}) - \Gamma(g') = \frac{1}{12\pi} \int_{S^3} d^3x \ \varepsilon^{ijk} \ \mathrm{tr} (\bar{g}^{-1} \partial_i \bar{g})(\bar{g}^{-1} \partial_j \bar{g}) \times (\bar{g}^{-1} \partial_k \bar{g})$$

where $\bar{g}$ is defined from $\tilde{g}, g'$ by joining a pair of 3-disks to a 3-sphere $S^3$.

It is another topological fact that this integral is equal to $2\pi \times$ integer for any smooth map $\bar{g} : S^3 \to G$. In the case of $G = SU(n)$ in the defining $n$-dimensional representation the basic unit $= 1$, that is, the value of the integral can be $2\pi m$ for any $m \in \mathbb{Z}$, by a suitable choice of $\bar{g}$.

<u>Exercise</u> Let $G = SU(2)$ in the defining representation and think of $S^3$ as the 1-point compactification of $\mathbb{R}^3$. Write

$$g(\vec{x}) := e^{i f(r) \vec{x} \cdot \vec{\sigma} / r}$$

where $r = |\vec{x}|$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are the Pauli matrices, and $f(r)$ is a real valued smooth function with $f(0) = 0$, $f(\infty) = n \cdot \pi$ for some $n \in \mathbb{Z}$. Compute the value of

$$\frac{1}{12\pi} \int_{\mathbb{R}^3} \varepsilon^{ijk} \operatorname{tr} (g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g) .$$

<u>Variation</u> of $L(g)$:

$$\delta L = \frac{1}{2\lambda^2} \int d^2x \, (g^{-1}\delta g) \, \partial_\mu (g^{-1} \partial^\mu g)$$

$$- \frac{n}{4\pi} \int d^2x \, (g^{-1}\delta g) \, \varepsilon^{\mu\nu} \partial_\mu (g^{-1} \partial_\nu g)$$

where for the variation of $\Gamma(g)$ we have used Stokes' theorem and $\varepsilon^{01} = -\varepsilon^{10} = 1$, $\varepsilon^{00} = \varepsilon^{11} = 0$.

Setting $\delta L = 0$ for an arbitrary variation we get the Euler-Lagrange equations

$$0 = \left( \frac{1}{2\lambda^2} + \frac{n}{4\pi} \right) \partial_- (g^{-1} \partial_+ g) + \left( \frac{1}{2\lambda^2} - \frac{n}{4\pi} \right) \partial_+ (g^{-1} \partial_- g)$$

where $x_\pm = \frac{1}{\sqrt{2}} (x_0 \pm x_1)$.

We see that there is a simplification at the critical values $\lambda^2 = \pm 2\pi/n$ of the coupling constant $\lambda$. Let us fix now $\lambda^2 = +2\pi/n$, $n \neq 0$. Then the E-L equations become

$$\partial_-(g^{-1}\partial_+ g) = 0.$$

The solutions of these equations are

$$g(x_-, x_+) = A(x_-)\, B(x_+)$$

where $A, B$ are smooth functions of one real variable. Recall that the conformal group of $\mathbb{R}^{1,1}$ is $\mathcal{D}iff(\mathbb{R}) \times \mathcal{D}iff(\mathbb{R})$, where the diffeomorphisms are acting independently on $x_\pm$. Thus our system is <u>conformally invariant</u> in the case of the critical values $\lambda^2 = \pm 2\pi/n$, $n \neq 0$.

<u>The model on $S^1 \times \mathbb{R}$</u>

We shall next the conformally invariant case $\lambda^2 = 2\pi/n$. We shall make the additional assumption that the fields are <u>periodic in the space coordinate $x^1$</u>, that is, we replace $\mathbb{R}$ by the compactification $S^1$. Thus

$$g(x^0, x^1 + 2\pi) = g(x^0, x^1) \quad \forall\ (x^0, x^1).$$

This means that in the factorization $g = A(x_-)\,B(x_+)$, if $A(x_-)$ is any smooth function on the interval $0 \le x \le 2\pi$ then

$$B(x_+ + 2\pi) = A(x_- - 2\pi)^{-1} A(x_-) \cdot B(x_+)$$
$$= \Delta(x_-) \cdot B(x_+)$$

so when $A$ has period $\Delta^{-1}$ (with right multiplication) then $B$ has period $\Delta$ (with left multiplication). The smooth paths with a given period $\Delta$ form a manifold $\cong LG$, the smooth loop group of maps $S^1 \to G$. Using the freedom $A \mapsto Ah$, $B \mapsto h^{-1}B$ where $h \in G$ is a constant we may assume that $A(0) = \mathbb{1}$. In order to guarantee that all the derivatives of $g$ with respect to $x^1$ are also $2\pi$-periodic (after a right translation),

$$A'(2\pi)A(2\pi)^{-1} = A'(0)A(0)^{-1} = A'(0).$$

With the initial condition $A(0) = \mathbb{1}$ the function $A(x)$ is completely determined by $a(x) := A'(x)A(x)^{-1}$, taking values in the Lie algebra of $G$. Thus the set of solutions of the equations of motion for the WZW model on $S^1 \times \mathbb{R}$ can be identified as

$$M = L\underline{g} \times LG.$$

We take the attitude: The phase space of a classical hamiltonian system = the space of solutions of the time evolution equations. This makes sense because in hamiltonian mechanics the time evolution is determined by differential equations first order in time, so the solutions are determined by the initial conditions, that is, points in the usual phase space of the system.

Thus the phase space of the WZW model on $S^1 \times \mathbb{R}$ is $M$.

A hamiltonian system is determined by the phase space $M$ and a symplectic form $\omega$. In local coordinates $\omega = (\omega_{ij})$ with $i,j = 1,2,\dots,\dim M$ such that

1) $\omega_{ij} = -\omega_{ji}$

2) $\det(\omega_{ij}) \neq 0$

3) $\partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij} = 0 \quad \forall i,j,k.$

Give a pair $X = \sum X^i \partial_i$, $Y = \sum Y^i \partial_i$ of vector fields we define

$$\omega(X,Y) := \sum \omega_{ij} X^i Y^j$$

so $\omega(X, Y)$ is a smooth real valued function. Given any smooth $h : M \to \mathbb{R}$ there is a unique vector field $X_h$ such that

$$\omega(X_h, Y) = -Y \cdot h \qquad \forall Y;$$

this follows from (2) above. The <u>Poisson bracket</u> of two functions $f, g$ is

$$\{f, g\} := X_f \cdot g = \omega(X_f, X_g).$$

The bracket is antisymmetric and $\{f, gh\} = \{f,g\}h + g\{f,h\}$ by Leibnitz' rule. Let $(\omega^{ij})$ be the <u>inverse</u> of the matrix $(\omega_{ij})$. Then

$$\{f, g\} = -\sum \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

in terms of local coordinates.

<u>Darboux' Theorem</u> If $\omega$ is a symplectic form on a <u>finite-dimensional</u> manifold $M$ then in an open neighborhood of a given point $p \in M$ one can choose local coordinates $q^1, \dots, q^n, p^1, \dots, p^n$ ($\dim M = 2n$!) such that

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp^i, \quad \text{i.e.}$$

$$\omega\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p^j}\right) = -\omega\left(\frac{\partial}{\partial p^j}, \frac{\partial}{\partial q^i}\right) = \delta_{ij} \text{ and the}$$

other components $= 0$. Thus in Darboux coordinates

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p^i}\right).$$

Thus in Darboux coordinates $(\omega_{ij})$, $(\omega^{ij})$ are (locally) constants.

<u>Exercise</u> Using Darboux coordinates show that

$$X_{\{f,g\}} = [X_f, X_g].$$

Exercise Using the previous exercise show that

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0$$

for all functions $f,g,h$. [Poisson brackets define a Lie algebra]

Ex. 1 $M=\mathbb{R}^{2n}$ with coordinates $\{q^1,\dots,q^n,p^1,\dots,p^n\}$
Here globally

$$\omega=\sum_{i=1}^{n} dq^i\wedge dp^i$$

Ex. 2 $M=S^2$. We can select $\omega$ = area form on $S^2$. In spherical (non-Darboux!) coordinates

$$\omega=\sin\theta\, d\theta\wedge d\phi .$$

Then $X_\phi=\frac{1}{\sin\theta}\partial_\theta$ and $X_\theta=-\frac{1}{\sin\theta}\partial_\phi$, so

$$\{\theta,\phi\}=-\{\phi,\theta\}=\frac{1}{\sin\theta}\quad(\text{for } 0<\theta<\pi).$$

Given a hamiltonian function $h: M\to\mathbb{R}$ the hamiltonian equations of motion are defined by

$$\dot{x}^i(t)=\{x^i,h\}\qquad i=1,2,\dots,2n$$

in terms of local coordinates $x^i$. In particular, for Darboux coordinates we have

$$\dot{p}^i=-\frac{\partial h}{\partial q^i}\qquad \dot{q}^i=\frac{\partial h}{\partial p^i}.$$

Now

$$X_h=-\sum_{i=1}^{n}\left(\frac{\partial h}{\partial p^i}\frac{\partial}{\partial q^i}-\frac{\partial h}{\partial q^i}\frac{\partial}{\partial p^i}\right)$$

and the meaning of the hamiltonian equations is that they define a flow on $M$ in the direction of the vector field $X_h$.

Let us return to the WZW-model with $G=SO(n)$ in the defining $n$-dimensional representation.

We apply the general formalism to the infinite-dimensional manifold $M = L\underline{g} \times LG$. At $(\xi, g) \in M$ define

$$\omega_{(\xi,g)}((u,X),(v,Y)) := -\int_{S^1} \operatorname{tr} \xi(x)[X(x),Y(x)]\,dx + \frac{k}{2\pi}\int_{S^1} \operatorname{tr} X(x)Y'(x)\,dx + \int_{S^1} \operatorname{tr}\big(u(x)Y(x) - v(x)X(x)\big)\,dx$$

($k$ is a constant)

This is a (nonconstant) symplectic form on $M$.

For each $c \in L\underline{g}$ define the function

$$h_c : M \to \mathbb{R} \qquad h_c(\xi, g) := +\int \operatorname{tr} \xi(x)\, c(x)\,dx .$$

Let $V_c$ be the vector field $V_c = \left(+\frac{k}{2\pi}\dot{c} + [\xi, c],\ +c\right)$.

One can check from the definitions that

$$V \cdot h_c = \omega_{(\xi,g)}(V_c, V)$$

for any vector field $V$ on $M$ and thus $V_c$ is the hamiltonian vector field for the function $h_c$.

Next $V_{c_1} \cdot h_{c_2} = +\frac{k}{2\pi}\int \operatorname{tr} \dot{c}_1 c_2 + \int \operatorname{tr} \xi\, [c_1, c_2]$

using the cyclicity of the trace. Thus

$$\{h_{c_1}, h_{c_2}\} = V_{c_1} \cdot h_{c_2} = h_{[c_1,c_2]} - \frac{k}{2\pi}\int \operatorname{tr} c_1 \dot{c}_2\,dx .$$

The functions $h_c$ ($c \in L\underline{g}$) are called the classical currents.

Assume now $k \neq 0$. Define

$$H(\xi, g) := \frac{\pi}{k}\int \operatorname{tr} \xi(x)^2\,dx .$$

Then $X_H = \left(\dot{\xi},\ \frac{2\pi}{k}\xi\right)$ and so

$$\{H, h_c\} = X_H \cdot h_c = h_{\dot{c}}$$

Since $\frac{d}{dx}$ is the generator for rotations on the circle $S^1$, the hamiltonian flow corresponding to the hamiltonian $H$ = group of rotations on $S^1$.

Morally, one should view $M$ as the cotangent bundle $T^*LG$ of the configuration space $LG$; formally, this is achieved using the identification $L\underline{g}^* \cong L\underline{g}$ through the nondegenerate bilinear form

$$\langle X, Y \rangle = \int \mathrm{tr}\, X(x) Y(x)\, dx \quad \text{on } L\underline{g}.$$

However, the vector space dual of $L\underline{g}$ is much bigger than the space of maps $Y \longmapsto \langle X, Y \rangle$ for a smooth $X$, so one should view $L\underline{g}^*$ as some 'restricted' dual of $L\underline{g}$.

Up to this we have discussed the structure of the classical phase space of the WZW model. Next we want to quantize the model, using the general philosophy of geometric quantization:

Starting from a classical phase space $(M, \omega)$ the first step is prequantization. One constructs a complex line bundle $L$ over $M$ such that the curvature of the line bundle $= i\omega$.

A complex line bundle is defined by stating that sections of the line bundle are a collection of local smooth functions

$$\psi_\alpha : U_\alpha \longrightarrow \mathbb{C} \qquad U_\alpha \subset M \text{ open},$$

where $\{U_\alpha\}$ is an open cover of $M$, such that

$$\psi_\beta(x) = \psi_\alpha(x)\, g_{\alpha\beta}(x) \quad \text{for } x \in U_\alpha \cap U_\beta$$

where $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^\times$ are local transition functions obeying the consistency condition

$$g_{\alpha\beta}(x)\, g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \forall\, x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$$

with $g_{\alpha\alpha}(x) \equiv 1$. A trivial line bundle is defined by

$$g_{\alpha\beta}(x) = \eta_\alpha(x)\, \eta_\beta(x)^{-1}$$

where $\eta_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^\times$. In case of a trivial line bundle a section can be defined as a global function $\psi : M \to \mathbb{C}$, after a modification

$$\psi_\alpha(x) \longmapsto \psi_\alpha(x)\, \eta_\alpha(x)^{-1} \equiv \psi(x) \quad \text{for } x \in \mathcal{U}_\alpha .$$

A hermitean structure in $L$ is defined if $|g_{\alpha\beta}(x)| = 1 \quad \forall\, x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Then an inner product (for square integrable sections) can be defined as

$$\langle \psi, \phi \rangle = \int_M \overline{\psi(x)}\, \phi(x)\, dx ,$$

where $dx$ is an integration measure on $M$ [assuming that $M$ is oriented]; note that now

$$\overline{\psi_\alpha(x)}\, \phi_\alpha(x) = \overline{\psi_\beta(x)}\, \phi_\beta(x), \qquad x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta .$$

Finally, a (hermitean) connection on $L$ is defined by a collection of local 1-forms $A^{(\alpha)} = \sum_i A_i^{(\alpha)} dx^i$ [in local coordinates on $M$] such that

$$A_i^{(\beta)} = A_i^{(\alpha)} + g_{\alpha\beta}^{-1} \frac{\partial}{\partial x^i} g_{\alpha\beta} \qquad (*)$$

on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Note that $A_i^{(\alpha)}$ is purely imaginary for a hermitean connection.

The covariant derivative $\nabla_X \psi$ of a section $\psi$

in the direction of a vector field $X$ on $M$ is locally defined as

$$\nabla_X = \sum_i X^i \nabla_i \quad , \quad \nabla_i \Psi_\alpha(x) = \partial_i \Psi_\alpha(x) + A_i^{(\alpha)}(x) \Psi_\alpha(x) .$$

The curvature is defined by

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + F(X,Y) .$$

In local coordinates $F_{ij} = F(\partial_i, \partial_j)$ $= \partial_i A_j^{(\alpha)} - \partial_j A_i^{(\alpha)}$ and this does not depend on the index $\alpha$, by the formula (*) p. 41.

<u>Theorem</u> Given a symplectic form $\omega$ on $M$ a complex line bundle with curvature $F = i\omega$ exists if and only if $\omega$ is <u>integral</u> in the sense that

$$\int_\Sigma \frac{\omega}{2\pi} \in \mathbb{Z}$$

for any closed two dimensional surface $\Sigma \subset M$.

Any two complex line bundles with the same $\omega$ are isomorphic; more generally, a pair of complex line bundles $L, L'$ are isomorphic if $[\omega] = [\omega']$, that is,

$$\int_\Sigma \omega = \int_\Sigma \omega'$$

for any $\Sigma \subset M$.

Given a complex line bundle $L$ over $M$ with $F = i\omega$ for a hermitean connection $\nabla$, we can construct differential operators

$$T_f := -i \nabla_{X_f} - f .$$

Then $[T_f, T_g] = -[\nabla_{X_f}, \nabla_{X_g}] + i X_f \cdot g$

$$-iX_g \cdot f = -\nabla_{[X_f, X_g]} - F(X_f, X_g)$$

$$+iX_f \cdot g - iX_g \cdot f = -\nabla_{X_{\{f,g\}}} - i\omega(X_f, X_g) + 2i\{f,g\}$$

$$= -\nabla_{X_{\{f,g\}}} + i\{f,g\}$$

$$= -i\, T_{\{f,g\}} \, .$$

Note that the function $f \equiv 1$ corresponds to the linear operator $T_f \equiv \mathbb{1}$.

The last step is the choice of polarization; this leads from the prequantized system to a truely quantum system. The physical reason is that a wave function $\psi(p,q)$ in the phase space contains too many variables: We could localize $\psi$ with arbitrary precision both in the momenta $p_i$ and in the coordinates $q_i$, violating Heisenberg uncertainity principle.

The polarization is a way to split the coordinates $x^i$ to momenta $p^i$ and configuration space coordinates $q^i$. While in the case of $M = T^* \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ this is obvious in the general case of a symplectic manifold it is far from obvious.

Returning to the phase space $M = L\underline{g} \times LG$ of the WZW-model we require that the sections $\psi$ are (covariantly) constant in the $\xi$-coordinate (in $L\underline{g}$ directions). This is possible since the curvature $F = i\omega$ vanishes when both arguments are vectors in $L\underline{g}$ directions. Thus our Schrödinger wave functions will be sections in a line-bundle over $LG$, with curvature

$$F(X,Y) = \frac{k}{2\pi} i \int_{S^1} \mathrm{tr}\, X(x) Y'(x)\, dx \, ,$$

where again right invariant vector fields on $LG$ are identified as elements in $L\underline{g}$.

he classical observables $h_c \in C^\infty(M)$ are now placed by the linear operators $\hat{X} = -i\nabla_X$,

$$[\hat{X},\hat{Y}] = -[\nabla_X,\nabla_Y] = -\nabla_{[X,Y]} - F(X,Y)$$

$$= -i\,\widehat{[X,Y]} + \frac{k}{2\pi i}\int \mathrm{tr}\, X Y'\, dx \qquad (A)$$

ving the same commutator algebra, except for the ctor $-i$, as the Poisson bracket algebra of the $h_c$'s page 39.

he commutation relations (A) define a central extension the loop algebra $L\mathfrak{g}$ and is known as an ffine Lie algebra, a subclass of Kac-Moody lgebras. Writing

$$X = \sum_n X_n e^{inx}$$

n the circle $S^1$, where $X_n \in \mathfrak{g}$, we get

$$[\hat{X}_n, \hat{Y}_m] = -i\,\widehat{[X_n, Y_m]} + k\, m\, \delta_{n+m} \langle X_n, Y_m\rangle$$

with $\langle X_n, Y_m\rangle = \mathrm{tr}(X_n Y_m)$. Let $\alpha \in C^\infty(S^1)$.
Define

$$l_\alpha : M \longrightarrow \mathbb{R}, \quad l_\alpha(\xi, g) := -\frac{2\pi}{k}\int_{S^1} \alpha(x)\,\mathrm{tr}\,\xi(x)^2\, dx$$

We can check

$$\omega(X_\alpha, Y) = -Y\cdot l_\alpha \quad \forall\, Y$$

when $X_\alpha = \left(-(\alpha\xi)', \mp\frac{2\pi}{k}\alpha\xi\right)$ and we get

$$\{l_\alpha, l_\beta\} = X_\alpha \cdot l_\beta = +\frac{2\pi}{k}\int \beta(x)\cdot 2\,\mathrm{tr}\,\xi(\alpha\xi)'\, dx$$

$$= +\frac{2\pi}{k}\int 2\,\mathrm{tr}\,(\beta\alpha'\xi^2 + \beta\alpha\,\xi\xi')\,dx$$

$$= +\frac{2\pi}{k}\int \mathrm{tr}\,(\beta\alpha' - \alpha\beta')\,\xi^2\, dx \qquad \text{(integration by parts)}$$

$$= l_{\alpha\beta' - \beta\alpha'}$$

But the product $(\alpha, \beta) \longmapsto \alpha\beta' - \beta\alpha'$ is just the commutator of vector fields

$$\left[\alpha \frac{d}{dx}, \beta \frac{d}{dx}\right] = (\alpha\beta' - \beta\alpha') \frac{d}{dx}.$$

Thus the $l_\alpha$'s are interpreted as generators of the Lie algebra of $\mathrm{Diff}(S^1)$. Setting $l_n := -i\, l_\alpha$ for $\alpha = e^{inx}$, $n \in \mathbb{Z}$, we get

$$\{l_n, l_m\} = (n-m)\, l_{n+m},$$

the Witt algebra.

$$\{l_\alpha, h_c\} = X_\alpha \cdot h_c = -\int \operatorname{tr} c(x)\, (\alpha\xi)'(x)\, dx$$

$$= \int \operatorname{tr} \alpha\, \dot{c}(x)\, \xi(x)\, dx = h_{\alpha\dot{c}},$$

so $l_\alpha$ generates diffeomorphisms acting on the parameter $c(x)$. In particular, $l_0$ corresponding to $l_\alpha$ with $\alpha(x) \equiv 1$ generates rotations on the circle $S^1$.

Choose a normalized basis $\{T_1, \dots, T_N\}$ in $\underline{g}$,

$$\langle T_i, T_j \rangle := \operatorname{tr} T_i T_j = -\delta_{ij}$$

and define $h_a^n(\xi) := \int e^{-inx} \operatorname{tr} T_a\, \xi(x)\, dx$.

Then $\{h_a^n, h_b^m\} = \gamma_{ab}^c\, h_c^{n+m} + ink\, \delta_{n+m}\, \delta_{ab}$

where $[T_a, T_b] = \gamma_{ab}^c\, T_c$. Furthermore,

$$\{l_n, h_a^m\} = -m\, h_a^{n+m}.$$

Since $h_a^q$ give the Fourier modes of the function $\xi(x)$, we also have

$$l_n = \frac{-i}{2k} \sum_{p,a} h_p^a\, h_{n-p}^a.$$

This is the classical (nonquantized) form of the Sugawara formula.