I CONFORMAL TRANSFORMATIONS

To define conformality we need a (pseudo)-Riemann metric. Riemann metric is defined on a (smooth) manifold M. A manifold is a Hausdorff space with local coordinates $(x^1, \ldots, x^n)$ and smooth coordinate transformations

$$x'^i = x'^i(x^1, \ldots, x^n)$$

n is the dimension of the manifold M. It is assumed that for any point $p \in M$ there is a system of local coordinates covering some open neighborhood of p.

Ex. 1 The space $\mathbb{R}^{p,q}$ for $p, q = 0, 1, 2, \ldots$ This space is as a vector space the same as $\mathbb{R}^{p+q}$, but with the 'inner product'

$$\langle x, y \rangle := x_1 y_1 + \cdots x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$$

In general, pseudo-Riemann metric is defined locally by a symmetric nondegenerate tensor

$$g = g_{ij}(x) \qquad i, j = 1, 2, \ldots, n.$$

The signature (p, q) with $p + q = n$ is defined as follows: Since g is symmetric it can be diagonalized at any point x. The number of positive eigenvalues is p and the number of negative eigenvalues is q.

Local coordinate transformations: $x'^i = x'^i(x^1, \ldots, x^n)$

$$g'_{ij}(x') = \sum_{k,l} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}(x)$$

Ex. 2 $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

In spherical coordinates $(x,y,z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. These coordinates are singular at the poles $\theta = 0$, $\theta = \pi$. However, it is possible to select nonsingular coordinates at the poles such that the coordinate transformations are smooth (Exercise!)

$$\begin{cases} X_\phi := \frac{\partial}{\partial\phi}(x,y,z) = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \\ X_\theta := \frac{\partial}{\partial\theta}(x,y,z) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \end{cases}$$

Euclidean inner product: $\langle X_\phi, X_\phi\rangle = \sin^2\theta$

$\langle X_\theta, X_\theta\rangle = 1$, $\langle X_\phi, X_\theta\rangle = 0$

Metric tensor $g_{\phi\phi} = \sin^2\theta$, $g_{\theta\theta} = 1$, $g_{\theta\phi} = g_{\phi\theta} = 0$, positive definite.

Ex. 3 The group $SL(2,\mathbb{R}) = \{$real 2×2-matrices A with $\det(A) = 1\}$.

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \quad \text{with } a > 0.$$

$\dim SL(2,\mathbb{R}) = 3$

Tangent vectors: Consider smooth curves

$\gamma_1(t) := e^{t\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \cdot A$

$\gamma_2(t) := e^{t\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} A$ $\qquad \gamma_3(t) := e^{t\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot A$

$\dot\gamma_1(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot A$, $\dot\gamma_2(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot A$, $\dot\gamma_3(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A$

Declare an inner product for tangent vectors at the point A by using the right translation by A^{-1}, setting

$$\langle \dot\gamma_i, \dot\gamma_j \rangle := \operatorname{tr}(\dot\gamma_i A^{-1})(\dot\gamma_j A^{-1})$$

Now $g_{11} = -2$, $g_{12} = g_{21} = -1$, $g_{22} = 0$,
$g_{13} = g_{31} = 0 = g_{23} = g_{32}$, $g_{33} = 2$

The eigenvalues are $-1 \pm \sqrt{2}$ and 2, so the signature of this metric is $(2,1)$.

<u>Remark</u> As in the previous example, it is sometimes more convenient to compute the metric g_{ij}, not in the coordinate directions, but for some other choice of tangent vectors at a point $p \in M$.

In a smooth (locally defined) transformation $x \mapsto f(x)$ on M a tangent vector $u = (u^1, \dots, u^n)$ [in the coordinate bases] is sent to the vector

$$u'^i = \sum u^j \frac{\partial f^i}{\partial x^j}$$

at the point $f(x)$.

<u>Definition</u> A map f is <u>conformal</u> if there is a smooth function $\Omega(x) > 0$ such that

$$\sum \frac{\partial f^k}{\partial x^i} \frac{\partial f^l}{\partial x^j} g_{kl}(f(x)) = \Omega^2(x) \cdot g_{ij}(x)$$

for all i, j. That is, the inner product of the transformed tangent vectors u', v' at the point $f(x)$ is equal to $\Omega^2(x)$ times the inner product of u, v at the initial point x.
In particular, a conformal transformation preserves the angles of vectors,

$$\frac{\langle u', v' \rangle}{|u'| \cdot |v'|} = \frac{\langle u, v \rangle}{|u| \cdot |v|} = \cos \theta_{u,v}$$

with $|u| = + \sqrt{|\langle u, u \rangle|}$.

The function Ω is called the conformal factor of the transformation f.

Ex. 1 In the Euclidean space $\mathbb{R}^n$ any rotation and any translation is conformal with factor $\Omega = 1$. The dilatation $f(x) = \lambda x$ is conformal with factor $\Omega = \lambda$.

Ex. 2 The case $\mathbb{R}^2 \cong \mathbb{C}$, Euclidean metric. Write $z = x + iy \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be conformal for some open connected set $U \subset \mathbb{C}$. Write $f(z) = u(x,y) + i v(x,y)$ for real u, v. Denoting $u_x = \frac{\partial u}{\partial x}$ etc. the conformality condition in the Euclidean metric becomes

$$(*) \quad u_x^2 + v_x^2 = \Omega^2 = u_y^2 + v_y^2 \text{ and } u_x u_y + v_x v_y = 0.$$

Thus if f is holomorphic, i.e. the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

are satisfied then f is conformal provided that $\Omega^2 = u_x^2 + v_x^2 > 0$ or equivalently

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

which means that f is locally invertible (inverse function theorem). The same conclusion holds for antiholomorphic functions,

$$u_x = -v_y, \quad u_y = v_x.$$

Conversely: Suppose $f = u + iv$ is conformal. Then (u_x, v_x) and (u_y, v_y) are perpendicular vectors in $\mathbb{R}^2$ by the last eq. $(*)$, and of equal length by the first equations $(*)$.

But a vector perpendicular to (u_x, v_x) and of equal length has to be $\pm(v_x, -u_x)$, that is, f is either holomorphic or antiholomorphic.

<u>Remark</u> If we require that f in the example above is everywhere holomorphic / antiholomorphic and invertible, one can show that f has to be of the form
$$f(z) = \xi z + \tau \quad \text{for } \xi, \tau \in \mathbb{C}, \ \xi \neq 0.$$
These transformations form the conformal group of the Euclidean plane $\mathbb{R}^2 \cong \mathbb{C}$.

<u>Conformal transformations in $\mathbb{R}^{p,q}$, $n = p+q \geq 3$.</u>

Metric $\langle x, y\rangle = x_1 y_1 + \dots + x_p y_p - x_{p+1} y_{p+1} - \dots x_{p+q} y_{p+q}$

1. The group $O(p,q) := \{R \text{ linear} \mid \langle Rx, Ry\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\}$
Elements of $O(p,q)$ are by definition conformal with $\Omega = 1$.

2. The group $\mathbb{R}^n$ of translations $x \longmapsto x + u$, $u \in \mathbb{R}^n$.

3. The group $\mathbb{R}_+$ of dilatations $x \longmapsto \lambda x$, $\lambda > 0$. Conformal factor $\Omega = \lambda$.

4. Special conformal transformations. Let $b \in \mathbb{R}^n$,
$$f(x) := \frac{x - x^2 b}{1 - 2\langle x, b\rangle + x^2 b^2} \quad \text{with } x^2 = \langle x, x\rangle.$$
Note that $f = f_b$ is <u>singular</u> in the subset of points x with $1 - 2\langle x, b\rangle + x^2 b^2 = 0$.
Nevertheless, these singular transformations form a group with
$$f_b \circ f_{b'} = f_{b+b'}, \quad \text{that is,}$$

the group K^n of special conformal transformations is isomorphic to the abelian group $\mathbb{R}^n$.

<u>Theorem</u> Any conformal transformation on an open connected set in $\mathbb{R}^{p,q}$ $(n = p+q \geq 3)$ is a composition of an element in $O(p,q)$, a translation, a dilatation, and of a special conformal transformation. They form a group of dimension $= \frac{1}{2}(n+1)(n+2)$.

Conformal compactification of $\mathbb{R}^{p,q}$

Search for compact $N^{p,q}$ with embedding $i : \mathbb{R}^{p,q} \to N^{p,q}$ compatible with conformal transformations acting in a <u>nonsingular</u> manner on $N^{p,q}$.

$$N^{p,q} := \{0 \neq (x^0, \ldots, x^{n+1}) \in \mathbb{R}^{p+1,q+1} \mid (x^0)^2 + (x^1)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - \ldots - (x^{p+q})^2 - (x^{n+1})^2 = 0\} / \mathbb{R}^\times$$

$n = p+q$

where $\mathbb{R}^\times = \{\lambda \in \mathbb{R} \mid \lambda \neq 0\}$ acts on vectors $x \in \mathbb{R}^{p+1,q+1}$ by $x \mapsto \lambda x$.

Clearly $N^{p,q} = (S^p \times S^q)/\mathbb{Z}_2$,

choosing normalization $(x^0)^2 + \cdots (x^p)^2 = 1$ by the $\mathbb{R}^\times$-action; the remaining $\mathbb{R}^\times$-action is the multiplication by $\lambda = \pm 1$.

<u>Theorem</u> The map $i : \mathbb{R}^{p,q} \to N^{p,q}$,

$$i(x) := \left(\frac{1-x^2}{2}, x^1, \ldots, x^n, \frac{1+x^2}{2}\right) / \mathbb{R}^\times$$

is a dense embedding.

<u>Proof</u> If $x^0 + x^{n+1} \neq 0$ parametrizes a point in $N^{p,q}$ we may assume $x^0 + x^{n+1} = 1$, using the $\mathbb{R}^\times$-action.

$$0 = (x^0)^2 + |x|^2_{p,q} - (x^{n+1})^2$$
$$= (x^0)^2 + |x|^2_{p,q} - (1-x^0)^2$$

$$\Longrightarrow \quad x^0 = \tfrac{1}{2}\left(1 - |x|^2_{p,q}\right).$$

Thus the points in $N^{p,q} \setminus i(\mathbb{R}^{p,q})$ are parametrized by vectors with $x^0 + x^{n+1} = 0$, that is, a surface in $N^{p,q}$ of dimension $\dim N^{p,q} - 1$.

The map i is smooth and injective. $\square$

<u>Conformal transformations on $N^{p,q}$</u>

1. Lorentz transformations $O(p,q)$.

$x \in \mathbb{R}^{p,q}$: $x \longmapsto \Lambda x$, $\Lambda \in O(p,q)$

$(x^0, x, x^{n+1}) \longmapsto (x^0, \Lambda x, x^{n+1})$ maps the light-cone

$(x^0)^2 + (x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^{n+1})^2 = 0$ onto itself and commutes with $\mathbb{R}^\times$, thus descends to a map $N^{p,q} \to N^{p,q}$.

2. Translations $x \longmapsto x + c$, $c \in \mathbb{R}^{p,q}$.

$$t_c : (x^0, x, x^{n+1}) \longmapsto \Big(\left(1 - \tfrac{1}{2}c^2\right)x^0 - \langle c, x\rangle_{p,q} - \tfrac{1}{2}c^2 x^{n+1},$$
$$x^0 c + x + x^{n+1} c,\ \tfrac{1}{2}c^2 x^0 + \langle c, x\rangle_{p,q} + \left(1 + \tfrac{1}{2}c^2\right)x^{n+1} \Big)$$

By direct computation one checks that these

transformation preserve the quadratic form $(x^0)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^{n+1})^2$; since they are linear they descend to maps $N^{p,q} \longrightarrow N^{p,q}$.
Also

$$\hat{i}(x + c) = t_c(\hat{i}(x)).$$

Check: Choose $x^0 = \frac{1}{2}(1 - x^2)$, $x^{n+1} = \frac{1}{2}(1 + x^2)$!

3. Dilatations $x \longmapsto \lambda x$.

$$R_\lambda : (x^0, x, x^{n+1}) \longmapsto \left(\frac{1+\lambda^2}{2\lambda} x^0 + \frac{1-\lambda^2}{2\lambda} x^{n+1}, x, \frac{1-\lambda^2}{2\lambda} x^0 + \frac{1+\lambda^2}{2\lambda} x^{n+1}\right) \sim \left(\frac{1}{2}(1+\lambda^2)x^0 + \frac{1}{2}(1-\lambda^2)x^{n+1}, \lambda x, \frac{1}{2}(1-\lambda^2)x^0 + \frac{1}{2}(1+\lambda^2)x^{n+1}\right) \bmod \mathbb{R}^*$$

Again, $\hat{i}(\lambda x) = R_\lambda(\hat{i}(x))$, by direct computation.

4. Special conformal transformations $x \longmapsto \dfrac{x - x^2 b}{1 - 2\langle x, b\rangle + x^2 b^2}$

$$K_b(x^0, x, x^{n+1}) = \left((1 - \tfrac{1}{2}b^2)x^0 - \langle b, x\rangle_{p,q} + \tfrac{1}{2}b^2 x^{n+1}, \; x^0 b + x - x^{n+1} b, \; -\tfrac{1}{2}b^2 x^0 - \langle b, x\rangle_{p,q} + (1 + \tfrac{1}{2}b^2)x^{n+1}\right)$$

Setting $x^0 + x^{n+1} = 1$, $x^0 = \frac{1}{2}(1 + x^2)$ one observes that

$$\hat{i}\left(\frac{x - x^2 b}{1 - 2\langle x, b\rangle + x^2 b^2}\right) = K_b(\hat{i}(x))$$

Thus all conformal (singular) transformation on $\mathbb{R}^{p,q}$ extend to nonsingular transformations on the conformal compactification $N^{p,q} = (S^p \times S^q)/\mathbb{Z}_2$.

The complement $N^{p,q} \setminus \mathbb{R}^{p,q}$?

In the complement of $\mathbb{R}^{p,q}$ we have $x^0 + x^{n+1} = 0$.

First, let $x^0 \neq 0$. Then, modulo the $\mathbb{R}^*$-action, we may assume $x^0 = 1 = -x^{m+1}$. But

$$x^0 + x^2 - (x^{m+1})^2 = 0 \implies \underline{x^2 = 0}$$

which defines the light cone in $\mathbb{R}^{p,q}$. The missing points $x^0 = 0 = x^{m+1}$ correspond to

$$\{0 \neq x \in \mathbb{R}^{p,q} \mid x^2 = \} / \mathbb{R}^* = S^{p-1} \times S^{q-1},$$

which together with the light cone $x^2 = 0$, $x^0 = 1$ form a compactification of the light cone.

II On Lie algebras

The conformal group is an example of a Lie group. By definition, a Lie group is a group G which at the same time is a smooth manifold such that the maps

$$G \times G \to G, \ (a,b) \mapsto ab \quad \text{and} \quad G \to G, \ a \mapsto a^{-1}$$

are smooth.

Ex. 1 The vector space $\mathbb{R}^n$ is a Lie group with group 'multiplication' $(x,y) \mapsto x + y$.

Ex. 2 The group $GL(n, \mathbb{R})$ of ~~r~~ $n \times n$ matrices A with $\det(A) \neq 0$. The group multiplication is the usual multiplication of matrices.

Theorem Any closed subgroup of a Lie group is a Lie group.

Ex. 3 Let $n = p + q$. The group $O(p,q) \subset GL(n, \mathbb{R})$ of matrices A such that

$$\langle Ax, Ay \rangle_{p,q} = \langle x, y \rangle_{p,q} \quad \forall x, y \in \mathbb{R}^{p,q}$$

All finite dimensional Lie groups we shall meet are subgroups of $GL(n,\mathbb{R})$ [or $GL(n,\mathbb{C})$].

The Lie algebra of a Lie group is determined by the 1-dimensional subgroups:

Ex. 1 $G = GL(n,\mathbb{R})$. If X is any real $n\times n$ matrix it generates a 1-dim. subgroup

$$g(t) = e^{tX}, \quad g(t+s) = g(t)\,g(s),$$

and $X = \frac{d}{dt} g(t)\big|_{t=0}$. The linear space $\mathfrak{gl}(n,\mathbb{R})$ of all real $n\times n$-matrices X forms the Lie algebra of $GL(n,\mathbb{R})$.

Def. A Lie algebra is a vector space $\mathfrak{g}$ with a Lie product

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X,Y) \mapsto [X,Y]$$

s.t. 1) $[X,Y]$ linear in both arguments

2) $[X,Y] = -[Y,X]$

3) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$.

The last condition is the Jacobi identity.

In Ex. 1 $\mathfrak{gl}(n,\mathbb{R})$ is a Lie algebra with the matrix commutator

$$[X,Y] = XY - YX.$$

Ex. 2 $G = O(n) \subset GL(\ ,\mathbb{R})$ of orthogonal matrices A, $A^t A = \mathbb{1}$. If now $g(t) = e^{tX} \in O(n)$ then

$$\mathbb{1} = g(s)^t\, g(s) \implies$$

$$0 = \frac{d}{ds} g(s)^* g(s) \Big|_{s=0} = X^* + X,$$

that is, X is <u>antisymmetric</u>. Conversely, if $X^* = -X$ then

$$g(s)^* = (e^{sX})^* = e^{sX^*} = e^{-sX} = g(-s)$$
$$= g(s)^{-1} \Rightarrow g(s) \in O(n).$$

The Lie algebra $\underline{o}(n)$ of $O(n)$ consists of all antisymmetric real $n\times n$-matrices. The Lie product is again the matrix commutator; we check

$$[X,Y]^* = (XY - YX)^* = Y^* X^* - X^* Y^*$$
$$= (-Y)(-X) - (-X)(-Y) = -[X,Y]$$

so indeed $\underline{o}(n) \subset \underline{gl}(n, \mathbb{R})$ is a Lie subalgebra

<u>Ex. 3</u> The abelian group $\mathbb{R}^n$ can be thought of as a subgroup of $GL(n+1, \mathbb{R})$ in the following way. For $x \in \mathbb{R}^n$ set

$$A_x := \begin{pmatrix} 1 & & 0 & x^1 \\ & \ddots & & \vdots \\ 0 & & 1 & x^n \\ & & & 1 \end{pmatrix}$$

Then $A_x \cdot A_y = A_{x+y}$. The Lie algebra $\underline{g}$ of $\mathbb{R}^n$ is then realized as matrices

$$T_x := \begin{pmatrix} 0 & \cdots & 0 & x^1 \\ & & & \vdots \\ & 0 & & x^n \\ & & & 0 \end{pmatrix}$$

since $e^{T_x} = \mathbb{1} + T_x = A_x$.

Clearly $[T_x, T_y] = 0 \quad \forall\, x, y \in \mathbb{R}^n$.

Ex 4 $O(p,q) := \{ A \in GL(p+q, \mathbb{R}) \mid \langle Ax, Ay \rangle_{p,q} = \langle x, y \rangle_{p,q} \ \forall x,y \in \mathbb{R}^{p,q} \}$.

$n = p+q$

If $p=0$ or $q=0$ then $O(p,q) = O(n)$.

Set $\eta = \mathrm{diag}(\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, -1, \dots, -1}_{q})$. Then

$\langle x, y \rangle_{p,q} = x^t \eta y$ when x, y are thought of as column vectors.

$$\langle Ax, Ay \rangle_{p,q} = (Ax)^t \eta (Ay) = x^t (A^t \eta A) y$$
$$= x^t \eta y \ \forall x,y \in \mathbb{R}^n \iff \underline{A^t \eta A = \mathbb{1}}.$$

Setting $A = A(s) = e^{sX}$ we get

$$0 = \frac{d}{ds} (e^{sX})^t \eta (e^{sX}) \Big|_{s=0} = X^t \eta + \eta X.$$

Thus the Lie algebra $\underline{\mathfrak{o}}(p,q)$ consists of real $n \times n$ matrices X s.t. $\underline{X^t = -\eta X \eta}$ since $\eta^2 = \mathbb{1}$.

Exercise Check that $\underline{\mathfrak{o}}(p,q)$ is closed under commutators, $[X,Y] \in \underline{\mathfrak{o}}(p,q)$ when $X, Y \in \underline{\mathfrak{o}}(p,q)$.

In particular, $O(3,1)$ is the Lorentz group in relativity theory, $\dim O(3,1) = 6$. It contains the group of rotations $SO(3)$ in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and 3 independent 1-parameter subgroups of 'Lorentz boosts'.

In general (exercise), $\dim O(p,q) = \dim O(p+q) = \frac{1}{2}(p+q)(p+q-1)$.

Lie group actions

A Lie group G acts on a manifold M if there is a multiplication map $G \times M \to M$, $(g, x) \longmapsto gx$ such that

$(g_1 g_2)x = g_1(g_2 x)$ $\forall g_1, g_2 \in G$, $x \in M$ and $ex = x$ where $e \in G$ is the unit element; the function $(g, x) \longmapsto gx$ is assumed to be smooth.

Ex. 1 Both the orthogonal group $O(3)$ and the group of translations act in a natural way in the space $\mathbb{R}^3$.

Ex. 2 Any group G acts on itself by left-multiplication $(g, x) \longmapsto gx$, with $g, x \in G$.

Ex. 3 The group $SL(2, \mathbb{R})$ acts in the upper half-plane $\mathbb{C}_+ := \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ as

$$gz = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}).$$

We have $\operatorname{Im}(gz) = |\gamma z + \delta|^{-2} \operatorname{Im}(z) \cdot (\alpha\delta - \beta\gamma)$ $= |\gamma z + \delta|^{-2} \operatorname{Im}(z) > 0$ when $z \in \mathbb{C}_+$

Let $H \subset G$ be a closed subgroup. Set

$$G/H := \{ gH \mid g \in G \}$$

the set of left cosets with respect to H. We also denote $[g] = gH$ when H is given.

Ex. 1 When $G = \mathbb{Z}$ and $H = p\mathbb{Z}$ (the group of integers divisible by $p \in \mathbb{Z}$) then $G/H = \mathbb{Z}_p$, the set of integers modulo p. Here $\mathbb{Z}_p$ is finite, $|\mathbb{Z}_p| = p$. The group $\mathbb{Z}$ acts on $\mathbb{Z}_p$ by addition of integers, $n \cdot [x] = [x + n]$.

Ex. 2 If $k < n$ are positive integers then $\mathbb{R}^k \subset \mathbb{R}^n$ is a (commutative) subgroup and

$$\mathbb{R}^n / \mathbb{R}^k \cong \mathbb{R}^{n-k}.$$

Ex. 3 Let $SO(2) \subset SO(3)$ be the group of rotations in the xy-plane in $\mathbb{R}^3$. Then $SO(3)/SO(2) \cong S^2$. The North Pole $N = (0,0,1)$ is invariant under $SO(2)$, $gN = N \ \forall g \in SO(2)$. Thus we have a map $f: SO(3)/SO(2) \to S^2$ by $g\,SO(2) \xmapsto{f} g \cdot N$ for $g \in SO(3)$. On the other hand, any point $x \in S^2$ is reached by a rotation from N, thus f is onto. If $f(g_1) = f(g_2)$ then $g_1 N = g_2 N \Rightarrow g_2^{-1} g_1 N = N \Rightarrow g_2^{-1} g_1 \in SO(2) \Rightarrow$ $[g_1] = [g_2]$, thus f is $1-1$.

Ex. 4 Show that $\mathbb{C}_+ \cong SL(2,\mathbb{R})/SO(2)$. [Compare with Ex. 3 and take $z = i$ to correspond N above.]

$\sim$

Let us return to the case of $G = O(p+1, q+1)$, the conformal group of $\mathbb{R}^{p,q}$ $(p+q = n \geq 3)$ acting in a nonsingular manner on $(S^p \times S^q)/\mathbb{Z}_2 = N^{p,q}$.

Note that in the constructions 1–4 on pages 7–8 all the conformal transformations are indeed linear transformations on $\mathbb{R}^{p+1,q+1}$ such that they leave invariant the pseudo-metric

$$\langle x, y \rangle_{p+1,q+1} = x^0 y^0 + x^1 y^1 + \dots + x^p y^p - x^{p+1} y^{p+1} - \dots - x^{p+q+1} y^{p+q+1},$$

thus elements of $O(p+1, q+1)$. It contains

1. $O(p,q) \subset O(p+1, q+1)$, the transformations with $x_0' = x_0$, $x_{n+1}' = x_{n+1}$.

2. The subgroup of translations $\mathbb{R}^n$ which as $(n+2)\times(n+2)$-matrices are

$$t_c = \begin{pmatrix} 1-\frac{1}{2}c^2 & -\tilde{c}^t & -\frac{1}{2}c^2 \\ c & \mathbb{1}_n & c \\ \frac{1}{2}c^2 & \tilde{c}^t & 1+\frac{1}{2}c^2 \end{pmatrix}$$

where $c \in \mathbb{R}^n$ is written as a column vector, c^t is the corresponding row vector, and $\tilde{c} = (c^1, \dots, c^p, -c^{p+1}, \dots, -c^{p+q})$.

Denote by e_i the unit (column) vector in the i-direction in $\mathbb{R}^{p,q}$ $(i = 1, 2, \dots, p+q)$. Then

$$\frac{d\,t_c}{dc^i}\Big|_{c=0} = \begin{pmatrix} 0 & -\tilde{e}_i^{\,t} & 0 \\ e_i & 0_n & e_i \\ 0 & \tilde{e}_i^{\,t} & 0 \end{pmatrix} \equiv T_i$$

gives an element in the Lie algebra $\underline{o}(p+1, q+1)$.
On checks by direct computation that indeed

$$T_i^t = -\eta\, T_i\, \eta$$

where $\eta = \operatorname{diag}(+1, \dots, +1, -1, \dots, -1)$ is the metric tensor in $\mathbb{R}^{p+1,q+1}$.

3. The dilatations:

$$R_\lambda = \begin{pmatrix} \frac{1+\lambda^2}{2\lambda} & 0 & \frac{1-\lambda^2}{2\lambda} \\ 0 & \mathbb{1}_n & 0 \\ \frac{1-\lambda^2}{2\lambda} & 0 & \frac{1+\lambda^2}{2\lambda} \end{pmatrix}$$

Now the derivative at the unit element $R_1 = \mathbb{1}_{n+2}$ is

$$\frac{d}{d\lambda} R_\lambda\Big|_{\lambda=1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0_n & 0 \\ -1 & 0 & 0 \end{pmatrix} \equiv D$$

is again in the Lie algebra $\underline{o}(p+1, q+1)$,

$$D^t = -\eta\, D\, \eta .$$

4. Special conformal transformations as matrices

$$K_b = \begin{pmatrix} 1-\frac{1}{2}b^2 & -\tilde{b}^t & \frac{1}{2}b^2 \\ b & 1_n & -b \\ -\frac{1}{2}b^2 & -\tilde{b}^t & 1+\frac{1}{2}b^2 \end{pmatrix}$$

$$K_i := \frac{dK_b}{db^i}\Big|_{b=0} = \begin{pmatrix} 0 & -\tilde{e}_i^t & 0 \\ e_i & 0 & -e_i \\ 0 & -\tilde{e}_i^t & 0 \end{pmatrix}$$

with $\qquad K_i^t = -\eta K_i \eta$.

The set of matrices $\{K_i, T_i, D\}$ together with the Lie algebra of $O(p,q) \subset O(p+1, q+1)$ span the whole Lie algebra $\underline{o}(p+1, q+1)$: One can easily check that the above matrices are linearly independent and furthermore

$$\dim \operatorname{span}\{K_i, T_i, D\} + \dim \underline{o}(p,q)$$
$$= 2(p+q)+1 + \frac{1}{2}(p+q)(p+q-1) = \frac{1}{2}(p+q+2)(p+q+1)$$
$$= \dim \underline{o}(p+1, q+1).$$

<u>Theorem</u> The conformal group $O(p+1, q+1)$ acts transitively on $N^{p,q}$: There is a point $z \in N^{p,q}$ such that $O(p+1,q+1)\cdot z = N^{p,q}$.

<u>Proof.</u> Let us choose $z = (1, 0, \dots, 0, 1)/\mathbb{R}^\times$.

Then $t_c(z) = (1-c^2, 2c, 1+c^2)/\mathbb{R}^\times$

which shows that we can reach any point in $\mathbb{R}^{p,q} \subset N^{p,q}$ from z by translations t_c.

The complement $N^{p,q} \setminus \mathbb{R}^{p,q}$ is parametrized by $(x^0, x^1, \dots, x^n, x^{n+1})$ such that $x^0 + x^{n+1} = 0$.
First, take any element in the complement s.t. $x^0 \neq 0$. Scaling by the $\mathbb{R}^\times$-action we may assume $x^0 = 1$.

$$K_b\,(1, x, -1) = \left(1 - b^2 - \langle b, x\rangle_{p,q},\ x + 2b,\ -1 - b^2 - \langle b, x\rangle\right)$$

which is in $\mathbb{R}^{p,q} \subset N^{p,q}$ provided that $b^2 + \langle b, x\rangle \neq 0$; for any x (with $x^2 = 0$) we may always choose a vector s.t. this inequality holds.
Finally, if $x^0 = 0$,

$$K_b \cdot (0, x, 0) = \left(-\langle b, x\rangle_{p,q},\ x,\ -\langle b, x\rangle_{p,q}\right).$$

Thus any point in $N^{p,q} \setminus \mathbb{R}^{p,q}$ can be mapped to $\mathbb{R}^{p,q}$ by a conformal transformation. Since the subgroup of translations acts transitively on $\mathbb{R}^{p,q}$ we conclude that the $O(p+1, q+1)$ action is transitiven on $N^{p,q}$. $\square$

Actually, the above discussion shows that already the <u>connected component of identity</u> $SO_0(p+1, q+1)$ acts transitively on $N^{p,q}$.

From the definitions of the conformal transformations one sees that the origin $\iota(0) = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2})/\mathbb{R}^\times$ is invariant under the 4-parameter abelian group K of special conformal transformations and the 1-dim. group of dilatations $\mathcal{D}$, as well as the group of 'Lorentz transformations' $SO_0(p, q)$.
Together these generate a 11-parameter group W and

$$N^{p,q} \cong SO_0(p+1, q+1) / W$$

The case of $\mathbb{R}^{1,1}$

Let $f, g : \mathbb{R} \to \mathbb{R}$ a pair of smooth 1–1 maps.
Set $f_\pm : \mathbb{R}^2 \to \mathbb{R}$, $f_\pm(x,y) := f(x \pm y)$.
Then the map $F(x,y) := \frac{1}{2}\left(f_+(x,y) + g_-(x,y),\ f_+(x,y) - g_-(x,y)\right)$
is conformal: This is by direct computation applying the derivative

$$dF = \begin{pmatrix} \frac{\partial F_u}{\partial x} & \frac{\partial F_u}{\partial y} \\ \frac{\partial F_v}{\partial x} & \frac{\partial F_v}{\partial y} \end{pmatrix}$$

to vectors $\binom{a}{b}, \binom{a'}{b'} \in \mathbb{R}^{1,1}$ and showing

$$aa' - bb' = \Omega^2 \left\langle dF \cdot \binom{a}{b}, dF \cdot \binom{a'}{b'} \right\rangle_{1,1}.$$

Here F_u, F_v denote $F = (F_u, F_v)$.

Denoting $\mathcal{D}iff(M) = \{$smooth invertible mappings of M onto itself$\}$
we see that $\mathcal{D}iff(\mathbb{R}) \times \mathcal{D}iff(\mathbb{R})$ are conformal transformations of $\mathbb{R}^{1,1}$ in the manner described above.
This group is disconnected, it contains four components

$$\mathcal{D}iff_\pm(\mathbb{R}) \times \mathcal{D}iff_\pm(\mathbb{R})$$

where $\mathcal{D}iff_+(\mathbb{R})$ is the group of orientation preserving diffeomorphisms f of $\mathbb{R}$, $f' > 0$, and $f \in \mathcal{D}iff_-(\mathbb{R})$ is orientation reversing, $f' < 0$.
The orientation preserving diffeomorphisms of $\mathbb{R}^{1,1}$ consists of $\left(\mathcal{D}iff_+(\mathbb{R}) \times \mathcal{D}iff_+(\mathbb{R})\right) \cup \left(\mathcal{D}iff_-(\mathbb{R}) \times \mathcal{D}iff_-(\mathbb{R})\right)$.

The conditions that a map $\mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$, $(x,y) \mapsto (u(x,y), v(x,y))$ is conformal
are $u^2 - u^2 > 0$ (for $\Omega^2 > 0$) a

$$u_x = v_y,\ u_y = v_x \quad \text{or} \quad u_x = -v_y,\ u_y = -v_x .$$

Note the change of signs compared to the Cauchy-Riemann equations in the Euclidean case $\mathbb{R}^2$!

Using these equations it is straightforward to show that all conformal transformations of $\mathbb{R}^{1,1}$ are given by elements of $\mathcal{D}iff(\mathbb{R}) \times \mathcal{D}iff(\mathbb{R})$, as above.

In field theory applications one often replaces the real line by the compactification S^1 and so $\mathbb{R}^{1,1} \longmapsto S^1 \times S^1$ and the conformal group becomes

$$Conf(S^1 \times S^1) = \mathcal{D}iff(S^1) \times \mathcal{D}iff(S^1)$$

One can define the Lie algebra of $\mathcal{D}iff(S^1)$ as before, considering 1-parameter subgroups $t \longmapsto h_t \in \mathcal{D}iff(S^1)$,

$$h_t \circ h_s = h_{t+s} \qquad h_0 = \mathbb{1} .$$

Now $\frac{d}{dt} h_t(z)\Big|_{t=0}$ with $z \in S^1$ defines a field of tangent vectors on S^1; this field X can be thought of as a differential operator. If f is a smooth function on S^1 then with $z = e^{i\theta}$,

$$\frac{d}{dt} f(h_t(z))\Big|_{t=0} = \dot{h}_0(e^{i\theta}) \frac{d}{d\theta} f(\theta) \equiv X(\theta) \frac{d}{d\theta} f(\theta)$$

so $X = X(\theta) \frac{d}{d\theta}$.

Commutators: $[X, Y] = [X \frac{d}{d\theta}, Y \frac{d}{d\theta}] = (XY' - YX') \frac{d}{d\theta}$.

It is convenient to complexify the vector fields:

$$X = \sum_n \alpha_n e^{in\theta} \frac{d}{d\theta} \qquad \alpha_n \in \mathbb{C} .$$

In particular $L_n := -i e^{-in\theta} \frac{d}{d\theta}$ which actually can be extended to a holomorphic vector field on $\mathbb{C} \setminus \{0\}$:

$$L_n = z^{1-n} \frac{d}{dz} .$$

We have
$$[L_n, L_m] = L_n L_m - L_m L_n$$
$$= (n-m) L_{n+m}$$

The Lie algebra consisting of (finite) linear combinations of L_n is called the <u>Witt algebra</u>.

Thus the conformal Lie algebra of the Minkowskian metric space $S^1 \times S^1$ is a direct sum of two copies of Witt algebras.

In the Euclidean case $\mathbb{C} \cong \mathbb{R}^{2,0}$ we have seen that the conformal transformations are either local holomorphic or antiholomorphic functions; their derivatives give holomorphic or antiholomorphic vector fields. In the holomorphic case we can use the basis

$$L_n = z^{1-n} \frac{d}{dz}, \quad n \in \mathbb{Z}$$

as before in the Minkowskian case. These are defined everywhere in $\mathbb{C} \setminus \{0\}$. The antiholomorphic case is

$$\overline{L}_m = \bar{z}^{1-n} \frac{d}{d\bar{z}}$$

for $z = x + iy$, $\bar{z} = x - iy$. These define another copy of the Witt algebra,

$$[\overline{L}_n, \overline{L}_m] = (n-m) \overline{L}_{n+m}$$

with
$$[L_n, \overline{L}_m] = 0 .$$

III The Virasoro algebra

Let $\underline{g}$ be a Lie algebra and $\underline{a}$ a vector space which we consider as an abelian Lie algebra. Let

$$c : \underline{g} \times \underline{g} \to \underline{a}$$

be an antisymmetric bilinear map. Try to define a Lie bracket in $\underline{g} \oplus \underline{a}$:

$$[(X,\lambda),(Y,\mu)] := ([X,Y]_{\underline{g}},\ c(X,Y)).$$

This expression is antisymmetric and bilinear. Jacobi identity? Since

$$[(X,\lambda),[(Y,\mu),(Z,\nu)]] = ([X,[Y,Z]]_{\underline{g}},$$
$$c(X,[Y,Z]))$$
the Jacobi identity is equivalent to

$$c(X,[Y,Z]) + c(Z,[X,Y]) + c(Y,[Z,X]) = 0$$

$\forall\, X,Y,Z \in \underline{g}$. Any form c which satisfies this identity is called a $\underline{a}$-valued <u>2-cocycle</u> on $\underline{g}$. Thes 1-cocycles define <u>central extensions</u> of Lie algebras. The name comes from the fact that the elements $(0,\lambda)$ with $\lambda \in \underline{a}$ commute with everything in $\underline{g} \oplus_c \underline{a}$.

A cocycl c is a <u>coboundary</u>, $c = \delta b$, if there is a linear map $b : \underline{g} \to \underline{a}$ such that

$$c(X,Y) = b([X,Y]_{\underline{g}}).$$

If $c = \delta b$ then we can define $f : \underline{g} \oplus \underline{a} \to \underline{g} \oplus_c \underline{a}$,

$$f(X,\lambda) := (X,\ b(X) + \lambda) \quad \text{and then}$$

$$[f(X,\lambda),\ f(Y,\mu)] = ([X,Y],\ c(X,Y)) +$$

$$(\,[X,Y],\ b([X,Y])\,)\,) = f(\,[(X,\lambda),\ (Y,\mu)]_0\,)$$

Thus the algebras $\underline{g} \oplus_0 \underline{a}$ and $\underline{g} \oplus_c \underline{a}$ are isomorphic.

So we are really interested only on the space of cocycles c modulo all coboundaries δb; this is denoted by

$$H^2(\underline{g}, \underline{a}).$$

In case of the Witt algebra W we can define $c : W \times W \to \mathbb{C}$ by linearity and

$$c(L_n, L_m) := \frac{n}{12}(n^2-1)\,\delta_{n+m} \qquad n, m \in \mathbb{Z}$$

where $\delta_m = 1$ for $m=0$ and $\delta_m = 0$ for $m \neq 0$. The normalization $1/12$ is arbitrary at this point; if c is a cocycle then αc is also a cocycle for any $\alpha \in \mathbb{C}$.

__Theorem__ $\dim H^2(W, \mathbb{C}) = 1$; any cocycle is of the form $\alpha \cdot c + \delta b$ for some b and $\alpha \in \mathbb{C}$.

__Proof.__ See M. Schottenloher, p. 79. □

The easy part is to show that c is indeed a cocycle: This is done by straightforward computation of

$$c(L_n, [L_m, L_p]) + c(L_p, [L_n, L_m]) + c(L_m, [L_p, L_n]) = 0.$$

In quantum field theory the elements of W are represented by linear operators in a (dense domain of) Hilbert space.

The __real__ Fourier modes $\cos(n\theta)\frac{d}{d\theta}$, $\sin(n\theta)\frac{d}{d\theta}$ should then be skew-symmetric operators. But

$$\cos(n\theta)\frac{d}{d\theta} = -\frac{i}{2}(L_n + L_{-n})$$

$$\sin(n\theta)\frac{d}{d\theta} = -\frac{1}{2}(L_n - L_{-n})$$

so we get the unitarity relations $L_n^* = L_{-n}$,
so called because if X is an element of a Lie algebra then the group elements $\exp(tX)$ $(t \in \mathbb{R})$ are unitary iff $X^* = -X$.

The Witt algebra $\mathcal{W}$ extended to $\mathcal{W} \oplus_c \mathbb{C}$ using the cocycle c is called the Virasoro algebra, Vir.
[The Virasoro algebra was first invented by Richard Block in the case of ~~a~~ a number field of characteristic p (1966).]

Def. A representation $\rho : \mathrm{Vir} \to \mathrm{End}(V)$ in a vector space V is a highest weight representation if there is a vector $v_0 \in V$ s.t.

1) $\rho(L_0)v_0 = h v_0$ for some $h \in \mathbb{C}$
2) $\rho(Z)v_0 = z v_0$ for some $z \in \mathbb{C}$
3) $\rho(L_n)v_0 = 0$ for $n = 1, 2, 3, \ldots$
4) v_0 is a cyclic vector.

The last property means that V is spanned by polynomials of the operators $\rho(L_n)$ acting on v_0.
Here Z is the generator of the 1-dimensional center of Vir; the relation 2) simply means that

$$[\rho(L_n), \rho(L_m)]v_0 = (n-m)\rho(L_{n+m})v_0 + \frac{z}{12}\delta_{n+m} \cdot n(n^2-1) \cdot v_0 .$$

In a unitary representation $\rho(L_0)^* = \rho(L_0)$ and so h is real. Furthermore, in a highest weight repr.

$$L_0(L_{-n}v_0) = [L_0, L_{-n}]v_0 + L_{-n}L_0 v_0$$
$$= n L_{-n}v_0 + h L_{-n}v_0 = (n+h)L_{-n}v_0$$

[we drop the symbol ρ when there is no confusion)
and so together with 3) we see that h is the
<u>minimal eigenvalue</u> of L_0.

Later, L_0 gets the role of energy and therefore the unitary highest weight representations are called <u>positive energy representations</u>.

For any pair $(z, h) \in \mathbb{C}$ there is always a highest weight representation, the so-called <u>Verma module</u> $M(z, h)$.

To any Lie algebra $\underline{g}$ one can associate an associative algebra $\mathcal{U}(\underline{g})$, the <u>universal enveloping algebra</u> of $\underline{g}$. The elements of $\mathcal{U}(\underline{g})$ are formal polynomials in the basis vectors $\{X_1, X_2, \dots\}$ of $\underline{g}$, modulo the Lie algebra relations

$$XY \equiv YX + [X, Y] \qquad X, Y \in \underline{g}.$$

For this reason a basis of $\mathcal{U}(\underline{g})$ is given by ordered monomials

$$X_{i_1} X_{i_2} \cdots X_{i_p} \qquad i_1 \geq i_2 \geq \cdots \geq i_p ;$$

this is the Poincaré-Birkhoff-Witt theorem.
A basic property of $\mathcal{U}(\underline{g})$ is that there is a canonical injection $\theta : \underline{g} \to \mathcal{U}(\underline{g})$ s.t.

$$\theta([X, Y]) = \theta(X)\,\theta(Y) - \theta(Y)\,\theta(X).$$

Furthermore, representations $\rho : \underline{g} \to \mathrm{End}(V)$ are in 1-1 correspondence with representations $\tilde{\rho} : \mathcal{U}(\underline{g}) \to \mathrm{End}(V)$ s.t.

$$\begin{tikzcd} \underline{g} \arrow[d, "\theta"'] \arrow[dr, "\rho"] & \\ {} \arrow[r] & \mathrm{End}(V) \end{tikzcd}$$

is commutative,

Let $\underline{n} \subset \underline{g}$ be a subalgebra. Then

$$\mathcal{U}(\underline{g})\underline{n} = \{ uX \mid u \in \mathcal{U}(\underline{g}),\ X \in \underline{n} \} \subset \mathcal{U}(\underline{g})$$

is a <u>left ideal</u> in $\mathcal{U}(\underline{g})$. Let $V = \mathcal{U}(\underline{g}) / \mathcal{U}(\underline{g})\underline{n}$ be the quotient vector space. We have now a canonical action of $\underline{g}$ (and of $\mathcal{U}(\underline{g})$) on V,

$$w \cdot (u \bmod \mathcal{U}(\underline{g})\underline{n}) := wu \bmod \mathcal{U}(\underline{g})\underline{n}.$$

We apply this construction to the case $\underline{g} = \mathrm{Vir}$. We set here

$$\underline{n} = \Big\{ \sum_{n \geq 1} \alpha_n L_n \,\Big|\, \alpha_n \in \mathbb{C} \Big\} \oplus \mathbb{C}\cdot(L_0 - h) \oplus \mathbb{C}\cdot(Z - z)$$

and $v_0 := \mathbb{1} \bmod \mathcal{U}(\underline{g})\underline{n} \in V$. Then by construction,

$$L_n v_0 = 0 \text{ for } n \geq 1, \quad L_0 v_0 = h v_0, \quad Z v_0 = z v_0$$

and also v_0 is a <u>cyclic vector</u>, $V = \mathcal{U}(\mathrm{Vir}) v_0$.

This is the Verma module $M(z,h) = V$.

In general, a Verma module is <u>not irreducible</u>: There are nontrivial submodules $W \subsetneq V$, i.e., subspace such that $\mathcal{U}(\underline{g}) W \subset W$.

<u>Theorem</u> Let $W \subset M(z,h)$ be a submodule of the Verma module for $\underline{g} = \mathrm{Vir}$. Then W is a direct sum of weight spaces,

$$W = \bigoplus_{\lambda} W_\lambda \; ; \quad v \in W_\lambda, \quad L_0 v = \lambda v.$$

<u>Proof</u> Any vector in $M(z,h)$ is a linear combination of vectors

$$L_{-n_1} L_{-n_2} \cdots L_{-n_p} v_0$$

for $n_i \geq 1$. But using $[L_0, L_m] = -m L_m$ we observe that this vector is an eigenvector

for L_0 with eigenvalue $h + n_1 + n_2 + \dots + n_p$.
If now $v \in W$ for a submodule $W \subset M(z,h)$, then

$$v = \sum \alpha_{n_1 \dots n_p} L_{-n_1} \cdots L_{-n_p} v$$

is a finite linear combination of weight vectors corresponding to eigenvalues $\lambda_1, \dots, \lambda_N \geq h$. For any $1 \leq i \leq N$,

$$\frac{1}{a} \prod_{j \neq i} (L_0 - \lambda_j)\, v$$

gives the projection of v to the component in W_{λ_i}; here

$$a = \prod_{j \neq i} (\lambda_i - \lambda_j). \quad \square$$

<u>Theorem</u> For any (z,h) there is a maximal submodule $W \subset M(z,h)$ not containing the vectors $\mathbb{C}\cdot v_0$; $M(z,h)/W$ is then an irreducible highest weight module for $\underline{g} = Vir$.

<u>Proof</u> According to the previous theorem any submodule in $M(z,h)$ is a direct sum of weight spaces. If $W \neq M(z,h)$ then W cannot contain the maximal weight vector $\mathbb{C}\cdot v_0$, otherwise $W = M(z,h)$ since v_0 is cyclic. Now we can define W as the sum of <u>all</u> submodules in $M(z,h)$ not containing the vector v_0. Since the projection of each of them on $\mathbb{C}\cdot v_0$ is zero, the same is true for the sum, thus $W \neq M(z,h)$. Clearly W is maximal, not containing v_0, so

$$M(z,h)/W =: V(z,h)$$

is irreducible. $\square$

In general, $V(z,h)$ is not unitary.

<u>Theorem</u> $V(z,h)$ is unitary for

1. $c \geq 1,\ h \geq 0$

2. $c = c(m) = 1 - \dfrac{6}{m(m+1)}$, $m = 2, 3, \ldots$

$$h = h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

for integers $1 \leq p \leq q < m$. These are the only unitary repr.

<u>Proof</u> See the refs. [KR87], [FQS 86], [GKO 86] in Schottenloher's book. $\square$

Instead of proving the above theorem, we give some concrete constructions of unitary representations.

<u>Bosonic Fock space construction</u>

Consider first the complex vector space $\mathcal{F}_0$ consisting of all polynomials in the variables $x_1, x_2, \ldots$. Define an inner product by

$$(*) \quad \langle x_1^{n_1} \cdots x_p^{n_p}, x_1^{n_1} \cdots x_p^{n_p} \rangle = \frac{n_1! \, n_2! \cdots n_p!}{1^{n_1} \cdot 2^{n_2} \cdots p^{n_p}}$$

and a pair of <u>different</u> monomials are orthogonal.

Define the operators $a_n := \dfrac{\partial}{\partial x_n}$ for $n = 1, 2, \ldots$

and $a_{-n} := n x_n$ $(n = 1, 2, \ldots)$, $a_0 := \mu \cdot \mathbb{1}$

for $\mu \in \mathbb{R}$. Then

$$[a_n, a_m] = + n \cdot \delta_{n+m} \quad \forall\, n, m \in \mathbb{Z}.$$

<u>Lemma</u> $\langle v, a_m w \rangle = \langle a_{-m} v, w \rangle \ \forall v, w \in \mathcal{F}_0$.

<u>Proof</u> By a direct computation from $(*)$. $\square$

The vector $v_0 = 1 \in \mathcal{F}_0$ is the <u>vacuum vector</u>,

$$a_n v_0 = 0 \ \forall\, n > 0, \quad a_0 v_0 = \mu v_0.$$

Define the <u>normal ordering</u>

$$:a_n a_m: = \begin{cases} a_n a_m & \text{for } n \le m \\ a_m a_n & \text{for } n > m. \end{cases}$$

Note that $:a_n a_m: = a_n a_m$ when $n \neq -m$ but

$$:a_n a_{-n}: = a_n a_{-n} - n \cdot \mathbb{1} \quad \text{for } n > 0.$$

Next define $\quad \rho(L_n) := \frac{1}{2} \sum_{k \in \mathbb{Z}} :a_k a_{n-k}:$

We denote in short $L_n = \rho(L_n)$. Although the sum above is infinite, applying L_n to a finite polynomial $v \in \mathcal{F}_0$ gives only a finite number of nonzero terms, thus

$$L_n : \mathcal{F}_0 \to \mathcal{F}_0$$

is a well-defined operator. Note that

$$\begin{cases} L_0 = \frac{1}{2} a_0^2 + \sum_{k \ge 1} a_{-k} a_k \\ L_{2n} = \frac{1}{2} a_n^2 + \sum_{k \ge 1} a_{n-k} a_{n+k} \\ L_{2n+1} = \sum_{k \ge 0} a_{n-k} a_{n+k+1} \end{cases} \quad (A)$$

<u>Theorem</u> The operators L_n in the Fock space $\mathcal{F}_0$ satisfy

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{n}{12}(n^2-1)\delta_{n+m} \cdot \mathbb{1}.$$

<u>Proof</u> Let first $n \neq 0$. Then

$$[L_n, a_m] = \frac{1}{2} \sum_{k \in \mathbb{Z}} [a_{n-k} a_k, a_m]$$

$$= \frac{1}{2} \sum_k a_{n-k} [a_k, a_m] + \frac{1}{2} \sum_k [a_{n-k}, a_m] a_k$$

$= -\frac{1}{2} a_{n+m} \cdot m \mp \frac{1}{2} m \cdot a_{n+m} = -m\, a_{n+m}$.

A similar computation shows $[L_0, a_m] = -m\, a_m$.

To prove the statement we need to show a) $[L_n, L_m] v_0 = (n-m) L_{n+m} v_0 + \frac{n}{12}(n^2-1) v_0$ and b) $[[L_n, L_m], a_k] = (n-m)[L_{n+m}, a_k]$ $\forall\, n, m, k$. Note that for $n+m \neq 0$:

$$[L_n, L_m] = \sum_k \frac{1}{2} [L_n, a_{m-k} a_k]$$

$$= -\frac{1}{2} \sum_k (m-k)\, a_{m+n-k}\, a_k + \frac{1}{2} \sum a_{m-k} \cdot (-k)\, a_{n+k}$$

$$= -\frac{1}{2} \sum_k (m-k)\, a_{m+n-k}\, a_k - \frac{1}{2} \sum_k (k-n) a_{m+n-k}\, a_k$$

$$= \frac{1}{2} \sum_k (n-m)\, a_{m+n-k}\, a_k = (n-m) L_{n+m}.$$

For this reason, to check (a) it remains the case $n+m=0$. For example, when $n>0$:

$$[L_n, L_{-n}] v_0 = (L_n L_{-n} - L_{-n} L_n) v_0$$

$$= L_n L_{-n} v_0 \qquad (\text{since } L_n v_0 = 0 \text{ for } n>0)$$

$$= \frac{1}{2} L_n \sum_{0 \geq k \geq -n} a_k\, a_{-n-k}\, v_0$$

$$= \frac{1}{2} \sum_{0 \geq k \geq -n} [L_n, a_k]\, a_{-n-k} v_0 + \frac{1}{2} \sum_{0 \geq k \geq -n} a_k [L_n, a_{-n-k}] v_0$$

$$= \frac{1}{2} \sum_{0 \geq k \geq -n} (-k)\, a_{n+k}\, a_{-n-k} v_0 + \frac{1}{2} \sum_{0 \geq k \geq -n} a_k \cdot (n+k) a_{-k}\, v_0$$

$$= n \cdot a_0^2\, v_0 + \frac{1}{2} \sum_{0 \geq k \geq -n+1} (-k)\, a_{n+k}\, a_{-n-k}\, v_0$$

$$= n a_0^2 v_0 - \frac{1}{2} \sum_{0 \geq k \geq -n+1} k(n+k)\, v_0$$

$$= n \mu^2 v_0 + \frac{1}{12} n (n^2 - 1)\, v_0$$

$$= \left[2n L_0 + \frac{1}{12} n(n^2-1) \right] v_0 .$$

The property (b): Use Jacobi identity,

$$[[L_n, L_m], a_k] = [L_n, [L_m, a_k]] - [L_m, [L_n, a_k]]$$

$$= [L_n, -k a_{m+k}] - [L_m, -k a_{k+n}]$$

$$= k(m+k)\, a_{n+m+k} - k(k+n)\, a_{n+m+k}$$

$$= k(m-n)\, a_{n+m+k} = \left[(n-m) L_{n+m} + \frac{n}{12}(n^2-1)\, \delta_{n+m} \cdot \mathbb{1} , a_k \right]$$

$\square$

Using the eq. (A) and the relation $(a_m)^* = a_{-m}$ we conclude that the representation of Vir in the Fock space $\mathcal{F}_0$ is unitary. In this case $z = 1$ and $h = \frac{1}{2}\mu^2$.

The inner product space $\mathcal{F}_0$ can be completed to a Hilbert space $\mathcal{F} \supset \mathcal{F}_0$. The operators L_n are unbounded and are not defined everywhere in $\mathcal{F}$.

Fermionic Fock space construction

Consider an algebra $\mathcal{B}$ generated by basis elements $\{b_m\}$ with

$$[b_n, b_m]_+ := b_n b_m + b_m b_n = \delta_{n+m} ;$$

The fermionic Fock space $\mathcal{F}_0$ is spanned by the vectors

$$v(n) = b_{m_1} b_{m_2} \cdots b_{m_p} v_0 \quad \text{with} \quad m_1 < m_2 < \dots m_p < 0 \quad \cancel{n_1 > n_2 > \dots > n_p} > 0$$

and defining relation $b_n v_0 = 0$ for $n \geq 0$.
The inner product is defined such that the basis $v(n) = v(n_1 \dots n_p)$ above is orthogonal and normalized.
One can then check

$$\langle v, b_k w \rangle = \langle b_{-k} v, w \rangle .$$

Next define $\quad L_n = -\frac{1}{2} \sum_{j} \left(j + \frac{1}{2} n\right) : b_{n+j} \, b_{-j} : + \alpha \cdot \delta_n$

where the fermionic normal ordering is $\qquad$ $\alpha =$ [illegible]

$$: b_n \, b_m : = \begin{cases} b_n \, b_m & \text{for } n \leq m \\ -b_m \, b_n & \text{for } n > m \end{cases}$$

Note that $: b_n \, b_m : = b_n \, b_m$ except for $n = -m > 0$,
$: b_n \, b_{-n} : = b_n \, b_{-n} - 1$ for $n > 0$.

Lemma $\quad [L_n, b_m] = -\frac{1}{2}(n + 2m)\, b_{n+m}$.

Proof $\quad [L_n, b_m] = -\frac{1}{2} \sum_{j} \left(j + \frac{1}{2} n\right) [b_{n+j} \, b_{-j}, b_m]$

$$= -\frac{1}{2} \sum_{j} \left(j + \frac{1}{2} n\right) b_{n+j} [b_{-j}, b_m]_+ + \frac{1}{2} \sum_{j} \left(j + \frac{1}{2} n\right) [b_{n+j}, b_m]_+ \, b_{-j}$$

$$= -\frac{1}{2}\left(m + \frac{1}{2} n\right) b_{n+m} + \frac{1}{2}\left(-m - n + \frac{1}{2} n\right) b_{n+m}$$

$$= -\frac{1}{2}(n + 2m)\, b_{n+m} . \quad \Box$$

for $\alpha = \frac{1}{16}$

Theorem $\quad [L_n, L_m] = (n-m) L_{n+m} + \frac{1}{24} n (n^2 - 1) \delta_{n+m} \cdot 1$

Proof $\quad$ As in the bosonic case, the interesting computation is $[L_n, L_{-n}] v_0$ for $n > 0$.
Again, from the definition of L_n it follows that

$$L_n v_0 = 0 \quad \text{for } n > 0 .$$

Thus $\quad [L_n, L_{-n}] v_0 = L_n L_{-n} v_0$

$$= L_n \cdot \left(-\frac{1}{2}\right) \sum_{j} \left(j - \frac{1}{2} n\right) : b_{-n+j} \, b_{-j} :$$

$$= L_n \cdot \left(-\tfrac{1}{2}\right) \sum_{n \geq j' > 0} \left(j' - \tfrac{n}{2}\right) b_{-n+j'} b_{-j'} v_0$$

$$= -\tfrac{1}{2} \sum_{n \geq j' > 0} \left(j' - \tfrac{n}{2}\right)\left(-\tfrac{1}{2}\right)(-n+2j') b_{j'} b_{-j'} v_0$$

$$+ \tfrac{1}{2} \sum_{n \geq j' > 0} \left(j' - \tfrac{n}{2}\right) b_{-n+j'} \cdot (n-2j') \cdot \left(-\tfrac{1}{2}\right) b_{n-j'} v_0$$

$$= \tfrac{1}{4} \sum_{n \geq j' > 0} \left(j' - \tfrac{1}{2} n\right)(-n+2j') v_0$$

$$= \tfrac{1}{2} \sum_{n \geq j' > 0} \left(j'^2 - j'n + \tfrac{1}{4} n^2\right) v_0$$

$$= \tfrac{1}{24} n (n^2 + 2) v_0 = \tfrac{1}{24} n(n^2-1) v_0 + 2n L_0 v_0 . \square$$

So we have a unitary representation $L_n^* = L_{-n}$ with $z = \frac{1}{2}$ and $h = 1/16$.