CONFORMAL FIELD THEORY, EXERCISE SET 8

1. Derive the formula $i[Q_k, \Phi_a(x)] = (x^2\partial_k - 2x_k E - 2\Delta_a x_k)\Phi_a$ for the generators Q_k of special conformal transformations, starting from the formula $x \mapsto x^b = (1+2x\cdot b + x^2b^2)^{-1}(x+x^2 b)$ for finite special conformal transformations. Here Φ_a is a field of conformal weight Δ_a and $E = \sum_k x^k\partial_k$.

2. Prove the formulas

$$
\begin{aligned}
[T, Y(a,z,\overline{z})] &= \partial_z Y(a,z,\overline{z}) \\
[H, Y(a,z,\overline{z})] &= (z\partial_z + \Delta_a)Y(a,z,\overline{z} \\
[T^*, Y(a,z,\overline{z}] &= (z^2\partial_z + 2\Delta_a z)Y(a,z,\overline{z})
\end{aligned}
$$

with $Y(a,z,\overline{z}) = (1+z)^{-2\Delta_a}(1+\overline{z})^{-2\overline{\Delta}_a}\Phi_a(t,\overline{t})$ in a two dimensional conformal field theory. Here $t = x_0 - x_1$ and $\overline{t} = x_0 + x_1$ and

$$
\begin{aligned}
P &= \frac{1}{2}(P_0 - P_1) \\
Q &= -\frac{1}{2}(Q_0 + Q_1) \\
T &= \frac{1}{2}(P + [P,Q] - Q) \\
H &= \frac{1}{2}(P + Q) \\
T^* &= \frac{1}{2}(P - [P,Q] - Q).
\end{aligned}
$$

3. Let $|0>$ be the vacuum vector in a 2d conformal field theory. Show that $H|a >= \Delta_a|a >$ and $T^*[a >= 0$ where $|a >= \Phi_a(t,\overline{t})|0>$ at $t = i, \overline{t} = i$ (in the light cone coordinates), that is, $z = 0 = \overline{z}$ for $z = \frac{1+it}{1-it}, \overline{z} = \frac{1+i\overline{t}}{1-i\overline{t}}$. Use the results of the previous exercise! Note that assuming the Wightman spectral condition for P_k and a similar condition for Q_k leads to $\Delta_a \geq 0$.

4. Prove the Wightman locality for the free real scalar field $\phi(x) = \phi^+(x) + \phi^-(x)$ (decomposition to positive and negative frequencies) in four space-time dimensions starting from $[\phi^+(x), \phi^-(y)] = \Delta^+(x-y)$ with

$$\Delta^+(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p_0}e^{-ip\cdot x}$$

with $p_0 = +\sqrt{\mathbf{p}^2 + m^2}$.