Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 5
10.10-15.10.2011

1. Suppose $(X, \mathcal{A})$ is a CW-complex and $(X_i, \mathcal{A}_i)$, $i \in I$ is a collection of subcomplexes of X. Prove that $(\bigcup_{i\in I} X_i, \bigcup_{i\in I} \mathcal{A}_i)$ and $(\bigcap_{i\in I} X_i, \bigcap_{i\in I} \mathcal{A}_i)$ are both subcomplexes of X.

   Solution: (The proof of) Lemma 4.1.4 implies that $(Y, \mathcal{B})$ is a subcomplex of $(X, \mathcal{A})$ if and only if
   1) $\mathcal{B} = \{\alpha \in \mathcal{A} \mid e_\alpha \cap Y \neq \emptyset\}$,
   and
   2) $e_\alpha \cap Y \neq \emptyset$ implies that $\overline{e}_\alpha \subset Y$.

   Now
$$e_\alpha \cap (\bigcup_{i\in I} X_i) \neq \emptyset \Leftrightarrow e_\alpha \cap X_i \neq \emptyset \text{ for some } i \in I \Leftrightarrow$$
$$\Leftrightarrow \alpha \in \mathcal{A}_i \text{ for some } i \in I \Leftrightarrow \alpha \in \bigcup_{i\in I} \mathcal{A}_i,$$
   and $e_\alpha \cap (\bigcup_{i\in I} X_i) \neq \emptyset$ implies that $e_\alpha \cap X_i \neq \emptyset$ for some $i \in I$ which implies that $\overline{e}_\alpha \subset X_i$ for some $i \in I$. In particular
$$\overline{e}_\alpha \subset \bigcup_{i\in I} X_i.$$
   Hence $(\bigcup_{i\in I} X_i, \bigcup_{i\in I} \mathcal{A}_i)$ is a subcomplex.

   Let us check the same for the intersection. Suppose $e_\alpha \cap \bigcap_{i\in I} X_i \neq \emptyset$. Then in particular $e_\alpha \cap X_i \neq \emptyset$ for all $i \in I$, so
$$\overline{e}_\alpha \subset X_i$$
   for all $i \in I$. Hence $\overline{e}_\alpha \subset \bigcap_{i\in I} X_i$. In particular condition 2) above is satisfied and if $\alpha \in \{\alpha \in \mathcal{A} \mid e_\alpha \cap \bigcap_{i\in I} X_i \neq \emptyset\}$, then $\alpha \in \bigcap_{i\in I} \mathcal{A}_i$.
   Conversely suppose $\alpha \in \bigcap_{i\in I} \mathcal{A}_i$. Then $\alpha \in \mathcal{A}_i$ for all $i \in I$, so
$$\overline{e}_\alpha \subset X_i$$
   for all $i \in I$, since $(X_i, \mathcal{A}_i)$ is a subcomplex for all $i \in I$. Hence
$$\overline{e}_\alpha \subset \bigcup_{i\in I} X_i.$$
   In particular $e_\alpha \cap \bigcup_{i\in I} X_i \neq \emptyset$, so $\alpha \in \{\alpha \in \mathcal{A} \mid e_\alpha \cap \bigcap_{i\in I} X_i \neq \emptyset\}$.

2. a) Suppose X is a CW-complex and A is a path-component of X. Prove that A is a subcomplex of X.

b) Suppose X is a CW-complex. Prove that the following claims are equivalent:
1) X is connected.
2) X is path-connected.
3) X^1 is path-connected.
4) Every two vertices in X^0 can be joined by a path that lies in X^1.

Solution: a) Suppose $e_\alpha \cap A \neq \emptyset$ and let $x \in e_\alpha \cap A$. Now $\overline{e}_\alpha$ is a path-connected subset of X (since it is a continuous image of the path connected space $\overline{B}^n$ for some $n \in \mathbb{N}$, which contains x. By the definition of path-component, $\overline{e}_\alpha \subset A$.
By Lemma 4.1.4 A is a subcomplex of A.

b) 1) $\Leftrightarrow$ 2):
All path-connected space are connected, so 2) implies 1) trivially.
Suppose X is a connected CW-complex and let $\mathcal{A}$ be the set of all path-connected components of X. If $\mathcal{A}$ is empty, X is empty, so it is trivially path-connected. Otherwise fix a path-component $A \in \mathcal{A}$ and define

$$B = \bigcup_{B \in \mathcal{A}, B \neq A} B.$$

Then $A \cup B = X$, $A \cap B = \emptyset$. A is a subcomplex by a), so it is in particular closed in X. Also every $B \in \mathcal{B}, B \neq A$ is a subcomplex for the same reason. By the exercise 1 B is a subcomplex, hence also B is closed in X. Hence B must be empty, since otherwise $A|B$ would be a separation of connected space X. Hence there is only one path-component A, which means that X is path-connected.

2)$\Leftrightarrow$ 3):
By Lemma 4.2.1c) the inclusion $i \colon X^1 \to X$ induces an isomorphism $i_* \colon H_0(X^1) \to H_0(X)$, hence in particular

$$H_0(X^1) \cong H_0(X)$$

Since a space Y is path-connected if and only if $H_0(Y) \cong \mathbb{Z}$, it follows that X is path-connected if and only if X^1 is path-connected.
3)$\Leftrightarrow$ 4):
Every point x of X^1 can be joined by the path to a vertex $a \in X^0$, since $x \in \overline{e}_\alpha$ for some $f_\alpha \colon \overline{B}^1$, where $\overline{e}_\alpha$ is path-connected and $\overline{e}_\alpha$ intersects X^0 (in subset $f_\alpha(S^0)$).
Hence X^1 is path-connected if and only if all vertices (i.e. points of X^0) belong to the same path-component of X^1.

3. Suppose K is a simplicial complex and a, b are vertices of K. **An edge-path** from a to b is a finite sequence of vertices $a = a_0, \ldots, a_n = b$ of K such that for all $i = 0, \ldots, n$ a_i and a_{i+1} belong to the same 1-simplex τ_i. In this case also the sequence $\tau_0, \ldots, \tau_{n-1}$ is also called an edge-path from a to b.
Prove that $|K|$ is connected if and only if for every pair of vertices $a, b \in K$

there is an edge-path from a to b.

Solution: Suppose $|K|$ is connected and let a be a vertex of a. Define subcomplex L, N of K as following. A simplex $\sigma \in K$ belongs to L if and only if all vertices of σ can be joined to a via an edge-path. A simplex $\sigma \in K$ belongs to N if and only if none of the vertices of σ can be joined to a via an edge-path.

Suppose $\sigma \in K$ is a simplex and $\sigma \notin N$. Then one vertex v of σ can be joined to a via edge-path $a = a_0, a_1, \ldots, a_n = v$. Let v' be any other vertex of σ. Then 1-simplex with vertices v and v' is a face of σ, so belongs to K. Hence $a = a_0, a_1, \ldots, a_n = v, a_{n+1} = v'$ is an edge-path from a to v'. Hence $\sigma \in L$, so $K = L \cup N$.
It follows that $|K| = |L| \cup |N|$. Clearly $|L| \cap |N| = \emptyset$. Since L, N are subcomplex of K, $|L|$ and $|N|$ are closed in $|K|$. Since $|L|$ is non-empty ($a \in |L|$), and $|K|$ is connected, N must be empty, since otherwise $|L|||N|$ is a separation of $|K|$.
In particular all vertices are in L, so every vertex can be joined to a by an edge-path.

Conversely suppose for every pair a, b of vertices there is an edge-path from a to b. Since edge-path clearly defines a continuous path from a to b in $|K|$, all vertices belong to the same path-component of $|K|$. Since every point $x \in |K|$ belong to some simplex $\sigma \in K$, which is path-connected and contains at least one vertex, every point belongs to the path-component of some vertex. These observations now easily imply that $|K|$ is path-connected, in particular connected.

4. Suppose $g \in \mathbb{N}$ ($g \geq 1$). Show that M_g (N_g) is a connected compact 2-manifold without boundary, which can be triangulated.

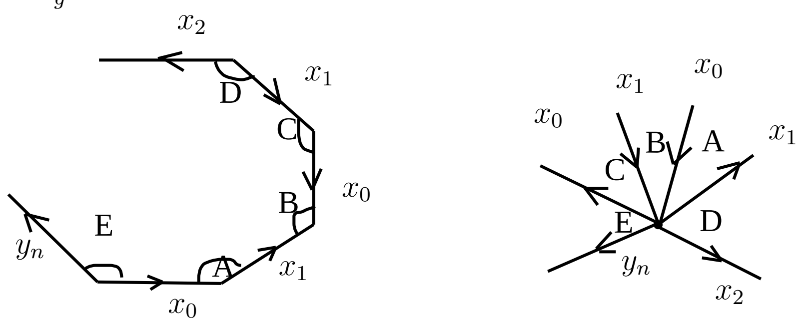
Solution: Clearly all these surfaces are compact and connected spaces, since they are quotient spaces of compact and connected polygon.

$M_0 = S^2$ and $N_1 = RP^2$ are known tobe 2-manifolds without boundary, so may assume $g \geq 1$ ($g \geq 2$).

Let X be $4g(2g)$-polygon and $p\colon X \to M_g(N_g)$ be a canonical quotient projection. Then the restriction of p to the interior of X is a homeomorphism to its image (for instance because by the exercise 11.6 it is a quotient mapping, which is also injective), which is also open in M_g (N_g) so the points in the image of interior have neighbourhoods homeomorphic to B_2.
Suppose x is an interior point of an edge τ in X. Then $p^{-1}(p(x)) = \{x, x'\}$, where x' is an interior point of another edge, identified with τ. Both have small enough neighbourhoods in X, which do not intersect other edges and are homeomorphic to $\{(x, y) \in B^2 \mid y \geq 0\}$, where the homeomorphism maps $\{(x, 0) \in B^2\}$ and only points in that set to the edge. Clearly both homeomorphisms can be chosen so that they can be fitted together as an embedding

$B^2 \to M_g(N_g)$ with image being an open neighbouhood of $M_g(N_g)$. Hence $p(x)$ has a neighbourhood homeomorphic to B^2.

We are left with the point that correspond to all vertices of the polygon. For every vertex we choose a small enough neighbourhood of a vertex, that do not contain any pairs of identified points, i.e. p is injection restricted to that neighbourhood. Then we stick them together as the picture below indicated for M_g.



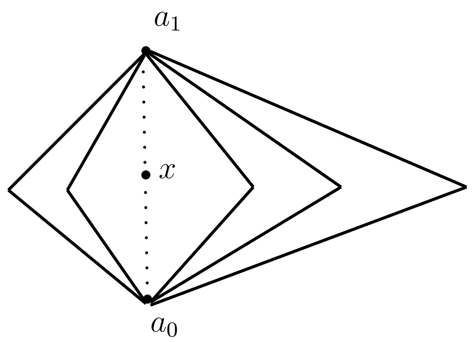
The case of a vertex for N_g is done the same way.

It remains to prove that M_g and N_g are trianguable. It is enough to triangulate them using some Δ-complex (). Choose a vertex a_0 in the interior of a polygon. Let $a_1, \ldots, a_m$ be vertices of the polygon. Then 2-simplices $[a_0, a_i, a_{i+1}]$, where $a_{n+1} = a_1$ and their faces form a triangulation of a polygon with ordering of simplices indicated by their indices. This triangulation defines a Δ-complex triangulation for any space obtained from a polygon by identifying edges in an obvious way.

5. Suppose K is a 2-dimensional simplicial complex and $\tau \in K$ is a 1-simplex which is a face of exactly n 2-simplices. Suppose x is an interior point of τ. Prove that

$$H_1(|Lk(x)|) \cong \mathbb{Z}^{n-1}.$$

Solution: $Lk(x)$ consists of all 1-faces (and their vertices) of simplices σ_i, that are not τ, so $|Lk(x)|$ is homeomorphic to the space obtained from n disjoint copies $I_1, I_2, \ldots, I_m$ of the unit interval $I = [0, 1]$ by identifying all points $0 \in I_i$, $i = 1, \ldots, m$ to a single point and also all points $1 \in I_i, i = 1, \ldots, m$ to a single point - see the picture below.



This space can be triangulated as a polyhedron $|K|$ of a Δ-complex K, that consists of n 1-simplices $\tau_1, \ldots, \tau_n$, with their corresponding vertices a, b

identified. We order every edge as a simplex $[a,b]$, then in $C_1(K)$

$$\partial\tau_i = b - a$$

for all $i = 1,\ldots,n$. Hence

$$\partial(m_1\tau_1 + m_2\tau_2 + \ldots + m_n\tau_n) = (m_1 + m_2 + \ldots + m_n)(b-a) = 0$$

if and only if $m_n = -m_1 - m_2 - \ldots m_{n-1}$. It follows that

$$\{\tau_1 - \tau_n, \tau_2 - \tau_n, \ldots, \tau_{n-1} - \tau_n\}$$

is a basis of the free abelian group $\operatorname{Ker}\partial_1$. Since there are no 2-simplices, for the simplicial homology we have

$$H_1(K) = \operatorname{Ker}\partial_1 \cong \mathbb{Z}^{n-1}.$$

The claim follows since the simplicial homology is isomorphic to the singular homology.

6. Suppose K is a finite simplicial complex such that $|K|$ is an n-dimensional manifold, possibly with boundary. Prove that $|Lk(x)|$ has the homotopy type of S^{n-1}, if $x \in |K|$ is an interior point and contractible if x is the boundary point.
Assuming $n = 2$ prove that $|K|$ is 2-dimensional as simplicial complex and every 1-simplex of K is a face of two or one 2-simplex. Moreover if L is a subcomplex of K generated by 1-simplices that are faces of exactly one 2-simplex, then $|L| = \partial|K|$.

Solution: For every $x \in |K|$ exists a homeomorphism $f\colon \overline{B}^n \to U$, U contains an open neighbourhood of x in $|K|$ and $f(y) = x$ if x is an interior point for $y = 0$, $f(y) = x$ for some $y \in S^{n-1}$ if x is a boundary point. Triangulate $\overline{B}^n$ as $|K(\sigma)|$, where σ is an n-simplex. If x is a boundary point, we may assume that y above is a vertex point of σ.
Now Proposition 4.3.5 implies that $|Lk(y)|$ and $|Lk(x)|$ have the same homotopy type. If x is an interior point $|Lk(y)|$ is homeomorphic to S^{n-1}. If x is a boundary point, $Lk(y)$ is a subcomplex generated by $n-1$-face of σ opposite to y, so $|Lk(y)|$ is contractible. This concludes the proof of the first claim.

Suppose $n = 2$. If K would have a maximal simplex σ of dimension $m > 2$, then the interior point of this simplex would have link homeomorhic to S^{m-1}. However S^{m-1} does not have the same homotopy type of S^{n-1} neither it is contractible. This is a contradiction with what we already proved. Hence $\dim K \leq 2$. Since K is not empty, there is at least one vertex a , and there has to be at least one 1-simplex that contains this vertex, since otherwise a would be an open discrete point in $|K|$, which is impossible for 2-manifold. Hence it is enough to prove that every 1-simplex of K is a face of two or one 2-simplex, since that would also imply that there is at least one 2-simplex, so $\dim K = 2$.

Let τ be 1-simplex and x be an interior point of τ. Suppose τ is a face of exactly m 2-simplices. Then previous exercise shows that $H_1(|Lk(x)|) \cong \mathbb{Z}^{m-1}$. On the other hand we already know that $H_1(|Lk(x)| \cong \mathbb{Z}$, if x is an interior point and $H_1(|Lk(x)) = 0$, if x is a boundary point. Hence $m-1 = 0, 1$, i.e. $m = 1$ or $m = 2$.

Suppose L is a subcomplex of K which consists of all 1-simplices of K, that are faces of exactly one 2-simplex and all their vertices. We claim that $|L| = \partial|K|$.
Suppose τ is a 1-simplex that is a face of exactly one 2-simplex $\sigma \in K$ and let x be an interior point of τ. Now x has clearly has a neighbourhood which is contained entirely in σ and does not intersect other faces of σ. Moreover we can choose this neighbourhood to be homeomorphic to $\{(x, y) \in B^2 \mid y \geq 2\}$. Hence $x \in \partial|K|$.
If, one the other hand x is a vertex in L, its arbitrary neighbourhood intersects interior of some 1-simplex of L, which we already proved to be contained in $\partial|K|$. Hence $x \in \overline{\partial|K|}$. But $\partial|K|$ is closed, so $x \in \partial|K|$. We have shown that $|L| \subset \partial|K|$.

To prove the opposite that x be a boundary point of $|K|$. There are 3-possibilities - x is a vertex, x is an interior point of 1-simplex or x is an interior point of 2-simplex. In the latter case x clearly has small neighbourhood homeomorphic to B^2, so x is not a boundary point. Suppose x is an interior point of 1-simplex τ. If τ is a face of 2 simplices, their union is homeomorphic to the square, with x being an interior point of square. Hence as above we see that x is an interior point of the manifold, contradicition. Hence τ faces only one triangle, so is an element of L. In this case $x \in |L|$.
We are left with the case x is a vertex. Let

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid x \in \sigma\}.$$

Then $\mathrm{St}(x)$ is an open neighbourhood of x (Lemma 1.2.7), that does not contain any other vertex of K, except x itself. Suppose all simplices of K that contain x are not in L. Then all points of $\mathrm{St}(x)$ except x are interior points of 2-simplices or 1-simplices that face two 2-simplices. But we already showed above that such points are not in the boundary of $|K|$. Hence $\mathrm{St}(x) \cap \partial|K| = \{x\}$, so x is a discrete point of $\partial|K|$, since $\mathrm{St}(x)$ was open. But this is not possible, since $\partial|K|$ is a 1-manifold (exercise 10.6). This contradiction shows that there is at least one 1-simplex in L that contains x, so $x \in |L|$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.