Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 11
Solutions

1. Let $n > 1$ and suppose $f\colon S^{n-1} \to S^{n-1}$ is a continuous mapping. Write $S^n = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x|^2 + |t|^2 = 1\}$ and define $\Sigma f\colon S^n \to S^n$ by the formula
$$\Sigma f(x,t) = \begin{cases} (|x| \cdot f(x/|x|), t), & \text{if } x \neq 0, \\ (x,t), & \text{if } x = 0. \end{cases}$$
Prove that Σf is continuous.

Solution: Continuity in the open subset $S^n / \setminus \{e_{n+1}, -e_{n+1}\}$ is clear, so it enough to show the continuity in the points $(0, \pm 1)$. This follows from the observation that
$$|\Sigma f(x,t) - \Sigma f(0, \pm 1)|^2 = |x|^2 \cdot |f(x/|x|)|^2 + |t - (\pm 1)^2 = |x^2| + |t - (\pm 1)|^2$$
clearly goes to zero when (x,t) approaches $(0, \pm 1)$.

2. Suppose $f\colon S^n \to S^n$ is **even**, i.e. $f(x) = f(-x)$ for all $x \in S^n$. Prove that $\deg f$ is an even integer and if n is even then $\deg f = 0$. (Hint: f factors through the projective space $\mathbb{R}P^n$).
For every $m \in \mathbb{Z}$ give an example of an even mapping $f\colon S^1 \to S^1$ with $\deg f = 2m$.

Solution: Since $f(x) = f(-x)$ for all $x \in S^n$, f factors through the projective plane $\mathbb{R}P^n$, i.e. there exists continuous $g\colon \mathbb{R}P^n \to S^n$ such that $f = g \circ \pi$, where $\pi\colon S^n \to \mathbb{R}P^n$ is a canonical quotient.
Hence in particular $f_* = \pi_* \circ g_*\colon H_n(S^n) \to H_n(S^n)$.
If n is even, $H_n(\mathbb{R}P^n) = 0$, hence $\pi_* = 0$ and so $f_* = 0$ i.e. $\deg f = 0$.

If n is odd $H_n(\mathbb{R}P^n) \cong \mathbb{Z} \cong H_n(S^n)$ and up to isomorphisms the mapping $\pi_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ is a homomorphism $\mathbb{Z} \to Z$, $n \mapsto 2n$. Hence if e is a generator of $H_n(S^n)$,
$$f_*(e) = \pi_*(g_*(e)) = \pm 2 g_*(e),$$
so $\deg f$ must be an even number.

Suppose $m \in N$ and let $p_{2m}\colon S^1 \to S^1$ be mapping $p(x) = x^{2n}$. Then p is even and $\deg p_{2m} = 2m$.

3. a) For every $x \in \overline{B}^n, x \neq 0$ let
$$\alpha(x) = 2\sqrt{\frac{1 - |x|}{|x|}}.$$

Define $h\colon \overline{B}^n \to S^n$ by

$$h(x) = \begin{cases} (\alpha(x)x_1, \alpha(x)x_2, \ldots, \alpha(x)x_n, 1-2|x|), & \text{if } x \neq 0 \\ e_{n+1} = (0, \ldots, 1), & \text{if } x = 0. \end{cases}$$

Prove that h is a well-defined continuous surjective mapping which restriction to B^n is a homeomorphism to $S^n \setminus \{-e_{n+1}\}$ and which maps S^{n-1} onto $-e_{n+1}$. Deduce that h induces a homeomorphism $\overline{B}^n/S^{n-1} \cong S^n$.

b) Define $f\colon S^n \to S^n$ so that $f|B_+ = h \circ g$, where g is a standard homeomorphism $B_+ \to \overline{B}^n$, $g(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ and $f|B_-$ is a constant mapping that maps everything to the south pole $-e_{n+1}$.
Prove that f is a well-defined continuous mapping and $f(x) \neq -x$ for all $x \in S^n$. Deduce that $\deg f = 1$.

Solution: a) First we need to check that f is well-defined, i.e. that $h(x) \in S^n$ for all $x \in \overline{B}^n$. This is a simple computation.

$$|h(x)|^2 = \alpha(x)^2|x|^2 + (1-2|x|)^2 = 4(1-|x|)|x| + (1 - 4|x| + 4|x|^2) = 1.$$

To prove continuity it is enough to check that h is continuous at 0. We have

$$|h(x) - h(0)|^2 = |h(x) - e_{n+1}|^2 = \alpha(x)^2|x|^2 + 4|x|^2 = 4((1-|x|)|x|2 + |x|^2,$$

which clearly goes to zero, when x approaches 0.
Suppose $z \in S^n \setminus \{e_{n+1}, -e_{n+1}\}$, i.e. $z_{n+1} \neq \pm 1$. Suppose $x \in \overline{B}^n$ is such that $h(x) = z$. Then $z_i = \alpha(x)x_i$, $i = 1, \ldots, n$ and $z_{n+1} = 1 - 2|x|$. Then

$$|x| = (1 - z_{n+1})/2 \in]0, 1[,$$

in particular

$$\alpha(x) = 2\sqrt{\frac{1-|x|}{|x|}} = 2\sqrt{\frac{1+z_{n+1}}{1-z_{n+1}}},$$

Hence

$$x = (z_1/2\sqrt{\frac{1+z_{n+1}}{1-z_{n+1}}}, \ldots, z_n/2\sqrt{\frac{1+z_{n+1}}{1-z_{n+1}}}).$$

On the other hand if we define x by that formula, we easily see that $x \in B^n \setminus \{0\}$ and $h(x) = z$. Hence $h^{-1}(z)$ is a singleton $\{x\}$.
Suppose $h(x) = e_{n+1}$, $x \neq 0$. Then

$$1 - 2|x| = 1,$$

hence $x = 0$, which is a contradiction. Hence $h^{-1}(e_{n+1})$ is also singleton $\{0\}$.
Finally suppose $h(x) = -e_{n+1}$. Then

$$1 - 2|x| = -1,$$

which implies that $|x| = 1$, i.e. $x \in S^{n-1}$. Conversely it is easy to see that $h(x) = -e_{n+1}$ for all $x \in S^{n-1}$. Hence $h^{-1}(-e_{n+1}) = S^{n-1}$.

In particular h is surjective. Moreover, since it is surjective, continuous mapping between a compact space and a Hausdorff space, h is a quotient mapping. By general topology (Topology II) h induces a homeomorphism $\overline{h}\colon \overline{B}^n/\sim_h \to S^n$, where $\sim_f$ is an equivalence relation defined by

$$x \sim_h y \iff h(x) = h(y).$$

But the calculations above show that $\overline{B}^n/\sim_h$ is exactly $\overline{B}^n/S^{n-1}$. Also we know that $h|B^n\colon B^n \to S^n \setminus \{-e_{n+1}\}$ is a continuous bijection. Exercise 10.4 implies that it is a homeomorphism (alternatively one can apply an exercise 6 below to h and open subset $S^n \setminus \{-e_{n+1}\}$ of S^n).

b) Suppose $x \in S^n$ is such that $f(x) = -x$. Suppose first $x \in B_+$. Since $f(e_{n+1}) = e_{n+1} \neq -e_{n+1}$, we may assume $x_i \neq 0$ for some $i = 1, \ldots, n$, so from $\alpha(x)x_i = -x_i$ we obtain $\alpha(x) = -1$, which is impossible.

On the other hand if $x \notin B_+$, then $x_{n+1} < 0$, while $f(x) = -e_{n+1}$, which last coordinate is -1, so there cannot be $f(x) = -x$.

By Lemma 3.7.3 f is homotopic to identity mapping, in particular $\deg f = \deg \mathrm{id} = 1$.

4. Suppose (X, A) is a topological pair and A is a closed subset of X. Let $f\colon A \to Y$ and let $p\colon X \sqcup Y \to X \cup_f Y$ be the canonical quotient projection. Then $p|X \setminus A$ is an open injection and $p|Y$ is a closed injection. In particular both restriction are embeddings, $p(X \setminus A)$ is open in $X \cup_f Y$ and $p(Y)$ is closed in $X \cup_f Y$.
Solution: Suppose U is open in $X \setminus A$. Then it is open in X (since $X \setminus A$ is open in X) and
$$p^{-1}(p(U)) \cap X = U,$$
$$p^{-1}(p(U)) \cap Y = \emptyset.$$
Hence $p(U)$ is open in $X \cup_f Y$. It follows that the restriction $p|X \setminus A\colon X \setminus A \to X \cup_f Y$ is an open mapping. It is also injective. Hence it is homeomorphism to its image, which is open in $X \cup_f Y$.

Suppose F is closed in Y. Then
$$p^{-1}(p(F)) \cap X = f^{-1}(F),$$
$$P^{-1}(p(U)) \cap Y = F.$$
Since f is continuous, $f^{-1}F$ is closed in A, and since A is closed in X, $f^{-1}F$ is closed in X. Hence $p(F)$ is closed in $X \cup_f Y$. It follows that the restriction $p|Y\colon YtoX \cup_f Y$ is a closed mapping. It is also injective. Hence it is homeomorphism to its image, which is closed in $X \cup_f Y$.

5. Suppose Z is obtained from Y by attaching n-cells. Show that the set of open cells depends only on the pair (Z, Y). Assuming Z is Hausdorff show that the same is true for closed cells.
Solution: The space $Z \setminus Y$ is a disjoint union of open n cells e_α, which are open in Z, hence also in $Z \setminus Y$. Hence $Z \setminus Y$ is a disjoint union of open sets e_α. Since e_α is a continuous image of an open ball B^n, it is connected. Hence $\{e_\alpha\}$ is exactly the collection of connected components of $Z \setminus Y$, which clearly

depends only on the pair (Z, Y).

From the general properties of continuous mappings we obtain

$$f(\overline{B}^n) \subset \overline{f}(B^n) = \overline{e_\alpha}.$$

On the other hand if Z is Hausdorff, $f(\overline{B}^n)$ is compact, hence closed, so contains closure of e_α. Hence closed cell $\overline{e}_\alpha$ is exactly the closure of the corresponding open cell e_α. It follows that the collection of closed cells coincides with the collection of topological closures of the components of $Z \setminus Y$, hence depends only on the set (Z, Y).

6. Suppose $p\colon X \to Y$ is a quotient mapping and $A \subset Y$ is open or closed. Show that $p|p^{-1}A\colon p^{-1}A \to A$ is a quotient mapping.

   Solution: Suppose A is open in Y. Since p is surjective, $p|p^{-1}A$ is surjective - every $a \in A$ has at least one inverse image x that maps to a, and then x belongs to $p^{-1}A$ by definition.
   Suppose $U \subset A$ is such that $p|^{-1}(U) = p^{-1}U$ is open in $p^{-1}A$. Since A is open in Y, $p^{-1}A$ is open in X, hence $p^{-1}(U)$ is open in X. Since p is a quotient mapping, U is open in Y. In particular it is open in A.

   The case A is closed is handled in the same way.

7. a) Suppose Z is obtained Y by attaching n-cells and C is a compact subset of Z. Then Z intersects only finitely many open cells of Z.
   b) Suppose X is a CW-complex and C is a compact subset of Z. Then there exists $n \in \mathbb{N}$ such that $C \subset X^n$.

   Solution: a) Choose a point $x_\alpha \in C \cap e_\alpha$ for every α such that $C \cap e_\alpha \neq \emptyset$. It is enough to show that the set

$$A = \{x_\alpha \mid C \cap e_\alpha \neq \emptyset\}$$

   is finite.
   Let F be any subset of A. Then F intersects Y in the empty set and the inverse image $f_\alpha^{-1}(F)$ is empty or a singleton for every α. Hence F is closed in Z. Hence every subset of A is closed in A, so A is discrete. On the other hand A is a closed subset of C, which is compact, so A is a compact discrete space. Hence A has to be finite.

   b) This is similar to a). Make a counter-assumption and construct an increasing infinite sequence

$$n_0 < n_1 < n_2 \ldots < n_k < n_{k+1}$$

   of integers, so that there exists a point $x_m \in C \cap (X^{n_m} \setminus X^{n_{m-1}}, m \geq 1$. Then the set

$$A = \{x_m \mid m \in \mathbb{N}\}$$

   is a subset of C, such that its every subset F is closed in X. This is because every closed n-cell does not intersect $X \setminus X^n$, so F intersects every closed cell in a finite, hence closed subset. Hence A is a compact discrete space. On the other hand by construction it is infinite. We obtain a contradiction.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.