Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 7
31.10-04.11.2011

1. a) Suppose X is a non-empty space and $x \in X$. For every path-component X_a of X which does not contain x choose a point $y_a \in X_a$. Prove that the set
$$\{[y_a - x] \mid a \in \mathcal{A}\}$$
is a basis for $\tilde{H}_0(X)$, which is thus a free abelian group.
Here $\mathcal{A}$ is a set of all path-components of X that do not contain x.

   b) Suppose $X = S^0 = \{1, -1\}$ is a 2-point discrete space. Show that $\tilde{H}_0(X) \cong \mathbb{Z}$ with $1-(-1)$ a generator and $\tilde{H}_n(X) = 0$ for $n \neq 0$.

2. Prove that Mobius band has the same homotopy type as S^1.

3. a) Suppose Y is a contractible space and X is any space. Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are continuous mappings. Prove that both f and g are homotopic to constant mappings. Also prove that Y is path-connected.

   b) Suppose Y is a non-empty space. Prove that the following conditions are equivalent:
1) Y is contractible.
2) The set $[X, Y]$ is a singleton for any space X.
3) Y is path-connected and the set $[Y, X]$ is a singleton for every non-empty path-connected space X.
4) Y has a homotopy type of a singleton space.
(Reminder: $[X, Y]$ is a a set of homotopy classes of mappings $f\colon X \to Y$).

4. a) Suppose $f\colon (X, A) \to (Y, B)$ is a mapping of pairs. Suppose that $f\colon X \to Y$ as well as $f|A\colon A \to B$ are homotopy equivalences. Prove that
$$f_*\colon H_n(X, A) \to H_n(Y, B)$$
is an isomorphism.

   b) Let
$$X = \bigcup_{n \in \mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}$$
(so-called "topological comb space ") and $x_0 = (0, 1)$. Prove that a constant mapping $f\colon (X, x_0) \to (x_0, x_0)$ is such that its restictions to $X \to x_0$ and $x_0 \to x_0$ are homotopy equivalences, but f is not a homotopy equivalence (as a mapping of pairs).

5. (Updated!) Suppose K is a **finite** Δ-complex. For every geometric n-simplex σ of K choose a point $x_\sigma \in \operatorname{int} \sigma$ and let $U = |K^n| \setminus \{x_\sigma | \sigma \in K_n/\sim\}$. Prove

that U is open in K^n and the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence.
Deduce that the inclusion $i\colon (|K^n|, |K^{n-1}|) \to (|K^n|, U)$ induces isomorphisms in relative homology in all dimensions.

6. Suppose C', C, D, D' are chain complexes, $f, g, h\colon C \to D$, $k, m\colon D \to D'$, $l\colon C' \to C$ are chain mappings.
a) Suppose H is chain homotopy from f to g, H' chain homotopy from g to h. Prove that $H + H'$ is a chain homotopy from f to h. Deduce that the relation "f and g are chain homotopic"is an equivalence relation in the set of all chain mappings $C \to D$.
b) Prove that $k \circ H$ is a chain homotopy from $k \circ f$ to $k \circ h$ and $H \circ l$ is a chain homotopy from $f \circ l$ to $g \circ l$.
c) Suppose H'' is a chain homotopy from k to m. Then $H'' \circ f + m \circ H$ and $k \circ H + H'' \circ g$ are chain homotopies from $k \circ f$ to $m \circ g$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.