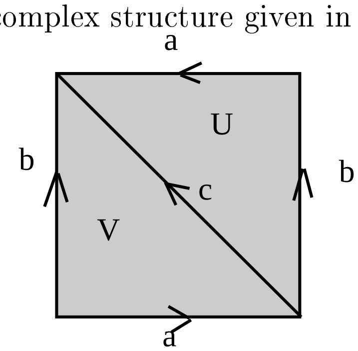
Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 5
Solutions

1. Consider S^1 as a polyhedron of a Δ-complex, generated by a single 1-simplex, whose vertices are identified. Calculate the simplicial homology of this complex. Verify that the result is the same as calculated in the lectures using the triangulation of S^1 as a boundary of a 2-simplex.

   Solution: Let us first make the following observation, which will also be useful while solving other exercises.
   Suppose K is a Δ-complex, that has only one (geometrical) vertex x. Then $\partial_1 \colon C_1(K) \to C_0(K)$ is zero mapping, hence
   $$H_1(K) = C_1(K)/\operatorname{Im}\partial_2 \text{ and}$$
   $$H_0(K) \cong C_0(K) = \mathbb{Z}[x] \cong \mathbb{Z}.$$
   Indeed in this case for every 1-simplex $[v,w] \in K$, we must have $v = w$, so
   $$\partial_1([v,w]) = w - v = 0.$$
   Now our Δ-complex has only one edge a and only one vertex x. Hence $H_n(K) = 0$ for $n \neq 0, 1$. Moreover by the observation above
   $$H_1(K) = C_1(K)/\operatorname{Im}\partial_2 = \mathbb{Z}[a]/\{0\} \cong \mathbb{Z} \text{ and}$$
   $$H_0(K) \cong \mathbb{Z}.$$
   Here we used the fact that there are no 2-simplices, so ∂_2 is a zero mapping as well.
2. Calculate the homology groups of the Klein bottle, using the familiar Δ-complex structure given in the picture below.

   Solution: In this complex there is only one vertex again, so
   $$H_0(K) \cong \mathbb{Z}$$
   and $H_1(K) = C_1(K)/\operatorname{Im}\partial_2$. Now $C_1(K) = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$, while
   $$\partial_2(U) = a + b - c = \alpha,$$
   $$\partial_2(V) = a - b + c = \beta.$$
   Now
   $$\partial_2(nU + mV) = n\alpha + m\beta = (n+m)a + (n-m)b + (m-n)c = 0$$

if and only if $n+m=n-m=0$, i.e. if and only if $n=m=0$. This shows that
1)∂_2 is an injection, hence $H_2(K)=0$, as a quotient of a trivial group $\operatorname{Ker}\partial_2$.
2) $\{\alpha,\beta\}$ is an independent set, which is then a basis of the free abelian group $\operatorname{Im}\partial_2$.

According to the exercise 4.6 $\{\alpha+\beta,\alpha\}=\{2a,a+b-c\}$ is also a basis for $\operatorname{Im}\partial_2$. On the other hand the application of this exercise to the set $\{a,b\}$ first implies that $\mathbb{Z}[a]\oplus\mathbb{Z}[b]=\mathbb{Z}[a]\oplus\mathbb{Z}[a+b]$, hence

$$C_1(K)=\mathbb{Z}[a]\oplus\mathbb{Z}[b]\oplus\mathbb{Z}[c]=\mathbb{Z}[a]\oplus\mathbb{Z}[a+b]\oplus\mathbb{Z}[c].$$

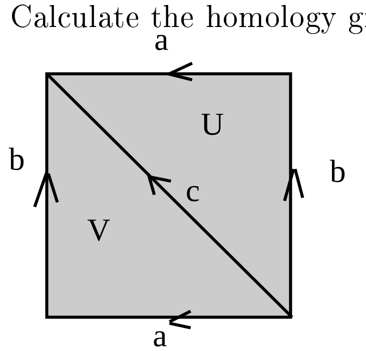
Another application of the exercise 4.6 shows that $\mathbb{Z}[a+b]\oplus\mathbb{Z}[c]=\mathbb{Z}[a+b-c]\oplus\mathbb{Z}[c]$, hence

$$C_1(K)=\mathbb{Z}[a]\oplus\mathbb{Z}[a+b-c]\oplus\mathbb{Z}[c].$$

Now it follows that

$$H_1(K)=(\mathbb{Z}[a]\oplus\mathbb{Z}[a+b-c]\oplus\mathbb{Z}[c])/(\mathbb{Z}[2a]\oplus\mathbb{Z}[a+b-c])\cong\mathbb{Z}[a]/2\mathbb{Z}[a]\oplus\mathbb{Z}[c]=\mathbb{Z}_2\oplus\mathbb{Z}.$$

3. Calculate the homology groups of the torus, using the Δ-complex structure



Solution: Once again there is only one vertex, so

$$H_0(K)\cong\mathbb{Z} \text{ and}$$

$$H_1(K)=C_1(K)/\operatorname{Im}\partial_2.$$

We have

$$\partial_2(U)=a+b-c=\partial_2(V).$$

Hence

$$\partial_2(nU+mV)=(n+m)(a+b-c)=0$$

if and only if $n=-m$. It follows that

$$H_2(K)=\operatorname{Ker}\partial_2=\mathbb{Z}[U-V]\cong\mathbb{Z}$$

and $\operatorname{Im}\partial_2$ is a free group $\mathbb{Z}[a+b-c]$.

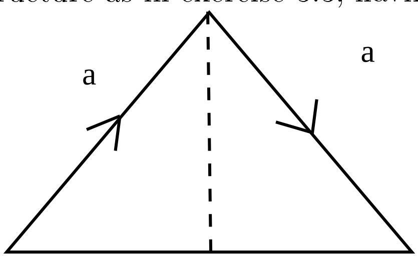
As in the similar exercise we easily see that

$$C_1(K)=\mathbb{Z}[a]\oplus\mathbb{Z}[b]\oplus\mathbb{Z}[a+b-c].$$

Hence

$$H_1(K)=(\mathbb{Z}[a]\oplus\mathbb{Z}[b]\oplus\mathbb{Z}[a+b-c])/(\mathbb{Z}[a+b-c])\cong\mathbb{Z}[a]\oplus\mathbb{Z}[b]\cong\mathbb{Z}\oplus\mathbb{Z}.$$

4. Calculate the homology groups of the Mobius band using the Δ-complex structure as in exercise 3.3, having one triangle with two sides indentified.



Solution: Again there is only one vertex, so

$$H_0(K) = \mathbb{Z}$$

and $H_1(K) = C_1(K)/\operatorname{Im}\partial_2$. There is only one 2-simplex U

$$\partial_2(nU) = n(2a - b),$$

which implies that ∂_2 is injection, hence $H_2(K) = 0$. Here $b = [v_0, v_2]$. Also $\operatorname{Im}\partial_2 = \mathbb{Z}[2a - b]$. Exercise 4.6 implies that $\{a - b, a\}$ is a basis for $C_1(K)$. Another application of the same exercise then implies that also $\{a-b+a, a\} = \{2a - b, a\}$ is a basis for $C_1(K)$. Hence

$$H_1(K) = (\mathbb{Z}[2a - b] \oplus \mathbb{Z}[a])/\mathbb{Z}[2a - b] \cong \mathbb{Z}[a] \cong \mathbb{Z}.$$

5. Find two short exact sequences of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C' \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

where C and C' are not isomorphic as groups.

Solution: Consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

where $f(n) = 2n$, $g(n) = n \bmod 2$ and the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f'} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{g'} \mathbb{Z}_2 \longrightarrow 0,$$

where $f'(n) = (n, 0)$ and $g'(x, y) = y$. Then sequences are exact. Clearly $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, for example because one of the groups is torsion free and the other is not.

6. Suppose H is a free abelian group, G is an abelian group and $f\colon G \to H$ is a surjective homomorphism. Prove that there is a homomorphism $f'\colon H \to G$ such that $f \circ f' = \mathrm{id}$.
Solution: Let $A \subset H$ be a basis of H. For every $a \in A$ choose an element $b_a \in G$ such that $f(b_a) = a$. This is possible, since f is a surjection. Now by the universal property of the basis there exists a (unique) homomorphism $f'\colon H \to G$ with the property $f'(a) = b_a$. Now for every $a \in A$ we have

$$f(f'(a)) = f(b_a) = a.$$

This implies that $f \circ f'$ coincides with identity on the elements of basis. This implies that $f \circ f' = \mathrm{id}$.

7. This exercise has moved to the next exercise session.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.