Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 4
Solutions

1. Let A be a set. For every $a \in A$ define $f_a \colon A \to \mathbb{Z}$ by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise .} \end{cases}$$

Prove that the set $\{f_a \mid a \in A\}$ is a basis of the abelian group $\mathbb{Z}^{(A)}$.

Solution: Suppose $g \in \mathbb{Z}^{(A)}$ and let $B = \{a_1, \ldots, a_k\}$ be its carrier. For $i = 1, \ldots, k$ denote $n_i = g(a_i)$. Then

$$g = n_1 f_{a_1} + \ldots + n_k f_{a_k} = \sum_{i=1}^{k} n_i f_{a_i}.$$

Hence $\{f_a \mid a \in A\}$ generates $Z^{(A)}$.

Suppose $a_1, \ldots, a_k \in A$ are different elements and $n_1, \ldots, n_k \in \mathbb{Z}$ are such that

$$g = n_1 f_{a_1} + \ldots + n_k f_{a_k} = \sum_{i=1}^{k} n_i f_{a_i} = 0.$$

Then

$$n_i = g(a_i) = 0$$

for all $i = 1, \ldots, k$.
Hence $\{f_a \mid a \in A\}$ is independent.

2. Prove that a free abelian group G is torsion-free i.e. for every $g \in G$ and $n \in \mathbb{N}$ the equation

$$ng = 0$$

is true if and only if $n = 0$ or $g = 0$.
Conclude that $\mathbb{Z}_n$, $n > 1$ or $\mathbb{Q}/\mathbb{Z}$ are not free abelian.

Solution: Let A be a basis of G. Suppose $x = n_1 a_1 + \ldots + n_k a_k \in G$, where $a_i \in A$ for all $i = 1, \ldots, k$, $n_i \in \mathbb{Z}$.
Now

$$nx = (nn_1)a_1 + \ldots + (nn_k)a_k = 0$$

if and only if $nn_i = 0$ for all i, since A is independent. If $n \neq 0$, this implies that $n_i = 0$ for all $i = 1, \ldots, k$. In this case $x = 0$.

$\mathbb{Z}_n$ is clearly not torsion-free, for $n > 1$, since $n\bar{1} = 0$ in $\mathbb{Z}_n$. Also $\mathbb{Q}/\mathbb{Z}$ is not torsion-free, since for example $n\overline{\frac{1}{n}} = 0$ in $\mathbb{Q}/\mathbb{Z}$.

3. Suppose $A \subset \mathbb{Q}$ contains at least 2 points. Show that A is not independent. Conclude that $\mathbb{Q}$ is not a free abelian group, although it is torsion-free.

Solution: Suppose $x, y \in A$, $x \neq y$. Then $x = a/c, y = b/d$, where $a, b, c, d \in \mathbb{Z}, c, d \neq 0$ and at least one of the numbers a or b is not zero.Now

$$(bc)x + (-ad)y = 0,$$

where at least one of the numbers bc or $-ad$ is not equal to zero. Hence A is not independent.

It follows that if $\mathbb{Q}$ is free abelian group, its basis has at most 1 element. It cannot have 0 elements, since then $\mathbb{Q}$ would be trivial group $\{0\}$. Suppose $\mathbb{Q}$ has a basis $\{x\}$, where $x = a/b$. We may assume that $a, b > 0$. Since $\{x\}$ is a basis of $\mathbb{Q}$, every element of $\mathbb{Q}$ equals nx for some $n \in \mathbb{Z}$. In particular it follows that there exists $n > 0$ such that

$$\frac{1}{b+1} = n\frac{a}{b}.$$

This implies that

$$b = na(b+1),$$

which is impossible. Hence $\mathbb{Q}$ is not free.

4. a) Let G be an abelian group and denote

$$2G = \{2g \mid g \in G\}.$$

Prove that $2G$ is a subgroup of G and show that if $G \cong H$, then $G/2G \cong H/2H$.
b) Suppose A and B are sets, A is finite. Prove that $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if B is finite and has the same amount of elements as A.
Solution: a) $2G$ is clearly non-empty. If $x = 2a, y = 2b \in 2G, a, b \in G$ then

$$x - y = 2(a - b)$$

(notice that commutativity of the addition is essential here). Hence $2G$ is a subgroup of G.
Suppose $f \colon G \to H$ is an isomorphism. Then clearly $f(2G) \subset 2H$ (since f is a homomorphism). By the same reasoning $f^{-1}(2H) \subset 2G$, i.e. $2H \subset f(2G)$. Hence

$$f(2G) = 2H.$$

Since f clearly induces isomoprhism $G/2G$ and $H/f(2G) = H/2H$, the claim is proved.

b) Suppose $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$. Then by a) also

$$\mathbb{Z}^{(A)}/2\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}/2\mathbb{Z}^{(B)}.$$

Now consider a mapping $h \colon \mathbb{Z}^{(A)} \to \mathbb{Z}_2^{(A)}$ defined to be a natural projection $\mathbb{Z}[a] \to \mathbb{Z}_2[a]$ on every summond $\mathbb{Z}[a]$ of $\mathbb{Z}^{(A)}$. h is clearly a surjection and its kernel is exactly $2\mathbb{Z}^{(A)}$. Hence there is an isomorphism $\mathbb{Z}^{(A)}/2\mathbb{Z}^{(A)} \cong Z_2^{(A)}$. Of course for the same reason $\mathbb{Z}^{(B)}/2\mathbb{Z}^{(B)} \cong Z_2^{(B)}$.

Hence if $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$, then also $\mathbb{Z}_2^{(A)} \cong \mathbb{Z}_2^{(B)}$. If A is a finite set, then $\mathbb{Z}_2^{(A)}$ is finite and contains exactly $2^|A|$ elements. If A is infinite, then $\mathbb{Z}_2^{(A)}$ is also infinite.
Hence it follows that if A is finite, also B must be finite and $2^|A| = 2^|B|$. Since the exponential function is an injection, this implies that $|A = |B|$.

The converse statement is almost trivial, so we skip the precise proof.

5. | a)Let G be a free group on 3 free generators a, b, c. Show that $\{c-a, b-a, a\}$ is also a basis of G.
b) Let G be a free group on 4 free generators a, b, c, d. Prove that the set $\{a+c+d, b-a+d, d\}$ is independent.
Solution: a) To show that the set $\{c-a, b-a, a\}$ generates the whole group it is enough to show that it generates at least the generators a, b, c. But
$$a = a,$$
$$b = (b-a) + a,$$
$$c = (c-a) + a.$$
It remains to show that the set is independent. Suppose
$$n(c-a) + m(b-a) + la = 0.$$
Then
$$(l-n-m)a + mb + nc = 0.$$
Since $\{a, b, c\}$ is independent, this implies that $m = n = l-n-m = 0$. Hence $n = m = l = 0$.

b) Suppose
$$n(a+c+d) + m(b-a+d) + ld = 0.$$
Then
$$(n-m)a + mb + nc + (n+m+l)d = 0.$$
Since $\{a, b, c, d\}$ is independent, it follows that
$$n-m = m = n = n+m+l = 0,$$
which implies easily that $n = m = l = 0$.

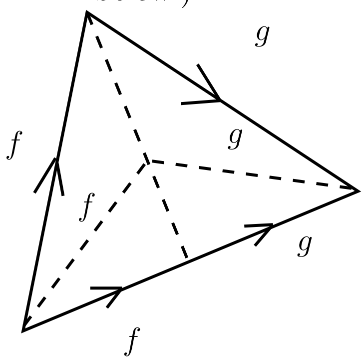
6. Suppose $\{\alpha, \beta\}$ is a basis of a group G. Prove that $\{\alpha \pm \beta, \beta\}$ is also a basis of G.
Solution: Since $\alpha = (\alpha \pm \beta) \mp \beta$, it follows that the set $\{\alpha \pm \beta, \beta\}$ generates G.
On the other hand if
$$n(\alpha \pm \beta) + m\beta = 0,$$
then $n\alpha + (m \pm n)\beta = 0$, which implies that $n = 0 = m \pm n$.

7. Suppose X is a topological space. Singular 1-simplices in X are mappings $f\colon I = [0,1] \to X$ and are also called **pathes** in X. If $f(0) = f(1)$ the path f is called **the loop**. Show that as an element of $C_1(X)$ the path f is a cycle if and only if it a loop.
Suppose $f, g\colon I \to X$ are pathes and $g(0) = f(1)$. Then we can define their **product** $f \cdot g\colon I \to X$ by

(continues on the other side)

$$(f \cdot g)(t) = \begin{cases} f(2t), & \text{if } 0 \le t \le 1/2, \\ g(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Prove that in this case $f + g - f \cdot g$ is a boundary element in $C_1(X)$ by constucting the explicit 2-simplex in X, whose boundary is $f+g-f \cdot g$. (Hint: see the picture below.)



Conclude that if f and g are loops, then

$$[f] + [g] = [f \cdot g] \in H_1(X).$$

Solution: Since

$$\partial(f) = f(1) - f(0),$$

it follows that $f\colon I \to X$ is a cycle if and only if $f(0) = f(1)$.

Suppose $f, g\colon I \to X$ and $f(1) = g(0)$. Define $H\colon \Delta_2 \to X$ by

$$H(x,y) = f \cdot g(y + x/2) = \begin{cases} f(2y+x), & \text{if } y \le -\frac{1}{2}x + \frac{1}{2} \\ g(2y+x-1), & \text{if } y \ge -\frac{1}{2}x + \frac{1}{2}. \end{cases}$$

Then it is easy to see that

$$\partial H = f + g - f \cdot g.$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.