Chapter 1

Simplicial stuff

1.1 Simplices

One of the main objectives of study in this course - **the singular homology theory** - is defined in terms of the continuous images of simplices. Simplices are also convenient for combinatorial representations of topological spaces and concrete computations of the algebraic invariants (such as the singular homology theory itself) in practise. This is why the first part of this course is dedicated to a brief introduction to simplices and simplicial methods.

We start off by recalling the notion of **a vector space** (over reals). It is a triple $(V, +, \cdot)$, where V is a set, $+\colon V \times V \to V$ is the operation of addition of vectors and $\cdot\colon \mathbb{R} \times V \to V$ is scalar multiplication. These operations are required to satisfy the " obvious " rules, in particular $(V, +)$ is a commutative group, every vector space has a zero vector 0 and so on. The mapping $f\colon V \to W$ between vector spaces is called linear if it preserves the addition and scalar multiplication i.e.

$$f(v + v') = f(v) + f(v'),$$
$$f(rv) = rf(v)$$

for all $v, v' \in V, r \in \mathbb{R}$.
In case you feel your knowledge of the basic properties of vector spaces and linear mappings is rusty, you should recall your basic Linear Algebra.

The canonical set of examples of a vector space is provided by **the n-dimensional Euclidean space** $\mathbb{R}^n$, where $n \in \mathbb{N}$. This space has a canonical basis $e_1, \ldots, e_n$ defined by

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 is the ith coordinate of e_i and the rest of the coordinates are 0's. Every vector $x \in \mathbb{R}^n$ can be represented as a linear combination

$$x = r_1 e_1 + \ldots + r_n e_n,$$

where $r_i \in \mathbb{R}$ in a unique way - in fact it is clear than this equation is true if and only if $x = (r_1, \ldots, r_n)$. The space $\mathbb{R}^n$ also has a canonical topological structure. We will also regard $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ via the identification

$$\mathbb{R}^n \ni (r_1, \ldots, r_n) = (r_1, \ldots, r_n, 0) \in \mathbb{R}^{n+1}.$$

We will mainly be interested in finite dimensional vector spaces. If $\{v_1, \ldots, v_n\}$ is a linear basis of a vector space V, then there is a unique linear isomorphism $f \colon V \cong \mathbb{R}^n$ such that $f(v_i) = e_i$, $i = 1, \ldots, n$. We can define a topology in V by requiring that f is a homeomorphism. It is easy to see that this topology does not depend on the choice of the basis (Exercise 1.1). In fact one can prove that this topology is a unique (Hausdorff) topology on V which makes it **a topological vector space**, i.e. such that the operations $+$ and $\cdot$ are continuous, but we won't be interested in this result. We will always regard a given finite-dimensional vector space V as a topological space equipped with this topology, which will be referred to as **the Euclidean topology** on V.

Let V be an (arbitrary) vector space. A subspace $A \subset V$ is called **affine** if

$$rx + (1-r)y \in A$$

for all $x, y \in A, r \in \mathbb{R}$. Geometrically this means that for every two (different) points of A the unique line that contains these points is contained in A. An empty set and every singleton $\{x\}, x \in V$ are thus trivially affine. In fact

Lemma 1.1.1. *Suppose $A \subset V$ is a non-empty affine subset. Then there is $v \in V$ and a linear subspace $W \subset V$ such that $A = x + W$. Moreover W is unique.*

Proof. Exercise 1.2. □

In other words affine sets are just translations of vector subspaces. In case V is finite-dimensional, also W is, hence, by uniqueness of W we can define affine dimension of A as $\dim A = \dim W$. For empty set one usually defines $\dim \emptyset = -1$.
Hence 0-dimensional affine spaces are singletons i.e. points, 1-dimensional -

lines, 2-dimensional - planes.

If we instead require a set to be closed only with respect to line segments between two points we obtain a very useful notion of a convex set. To be precise a subset $A \subset V$ is called **convex** if

$$rx + (1-r)y \in A$$

for all $x, y \in A, r \in [0,1]$. An affine set is necessarily convex. Every convex set is connected, even path-connected. A closed n-dimensional ball

$$\overline{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

is clearly not affine (why?), but it is convex (why?). The same is true for the open ball

$$B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

A punctured ball $\{x \in \mathbb{R}^n \mid 0 < |x| < 1\}$ with origin removed is an example of a connected set (for $n > 1$) which is not convex.
A mapping $f\colon C \to C'$ between convex sets is called **affine** if

$$f(rx + (1-r)y) = rf(x) + (1-r)f(y)$$

for all $x, y \in C, r \in [0,1]$. Also the term " linear mapping " is widely in use.

The intersection of an arbitrary collection of affine/convex sets is easily seen to be affine/convex. So for every subset $A \subset V$ there is the smallest affine set $\mathrm{aff}(A)$ that contains A and the smallest convex set $\mathrm{conv}(A)$ that contains A. The set $\mathrm{aff}(A)$ is called **the affine hull** of A and the set $\mathrm{conv}(A)$ -**the convex hull** of A. Clearly

$$A \subset \mathrm{conv}(A) \subset \mathrm{aff}(A).$$

There is also a simple, explicit way to express both hulls in terms of the points of A. Suppose $a_1, \ldots, a_n \in V$ are arbitrary points. A vector

$$r_1 a_1 + \ldots + r_n a_n = x \in V$$

is called an affine combination of the points $a_1, \ldots, a_n$ if $r_1 + \ldots + r_n = 1$. If also $r_i \geq 0$ for all i, it is called **a convex combination** of the points $a_1, \ldots, a_n$. By induction on n it is easy to see that affine/convex set is always closed under affine/convex combinations of its points (exercise 1.3a). More generally we have

Lemma 1.1.2. *Suppose $A \subset V$. Then*

$$\mathrm{aff}(A) = \{r_1a_1 + \ldots + r_na_n \mid a_i \in A, r_1 + \ldots + r_n = 1\},$$

$$\mathrm{conv}(A) = \{r_1a_1 + \ldots + r_na_n \mid a_i \in A, r_i \geq 0, r_1 + \ldots + r_n = 1\}.$$

Proof. Exercise 1.3b ☐

If V is finite dimensional, then every affine subset is an affine hull of a finite set of points - just translate it to a linear subspace, take a finite linear basis of this subspace and translate it back. These points will generate the affine set in question.
The situation is not that simple with convex sets. For example it is easy to see that a closed ball $\overline{B}^n$ cannot be a convex hull of a finite set if $n \geq 2$ (can you prove it?) and it is even easier to see the same for a corresponding open ball B^n, even for $n = 1$. In fact a convex hull of a finite set is always closed, even compact.

A convex hull of a finite set is usually called **a linear (closed) cell**. For example a square, more generally n-cube is a linear cell, so is a triangle or a pyramid with a triangle or square base. We won't need a general notion of a linear cell, so we will in fact restrict our attention to a useful special case of **the simplex**. To define the notion of simplex we first need a notion of affinely independent subset.

Consider a finite set $\{v_1, \ldots, v_n\}$ of points in V. We already know that every point x of the set $\mathrm{conv}\{v_1, \ldots, v_n\}$ can be written in the form

$$x = r_1v_1 + \ldots + r_nv_n,$$

where $r_i \geq 0$, $\sum_{i=1}^n r_i = 1$. In general a representation of x in this form is not unique. If it is always unique, we say that the set $\{v_1, \ldots, v_n\}$ is affinely independent. To be more precise let us first prove the following

Lemma 1.1.3. *Suppose $v_0, \ldots, v_n, \in V$. Then the following conditions are equivalent.*
a) $v_1 - v_0, \ldots, v_i - v_0, v_n - v_0$ is a linearly independent set of vectors.
b) If

$$\sum_{i=0}^{n} r_iv_i = 0 \text{ and } \sum_{i=0}^{n} r_i = 0,$$

then $r_i = 0$ for all $i = 0, \ldots, n$.
c) If

$$\sum_{i=0}^{n} r_iv_i = \sum_{i=0}^{n} r_i'v_i \text{ and } \sum_{i=0}^{n} r_i = \sum_{i=0}^{n} r_i',$$

then $r_i = r'_i$ *for all* i.

d) Every point in the affine hull $\operatorname{aff}(\{v_0, \ldots, v_n\})$ *has a unique representation in the form*

$$r_0 v_0 + \ldots + r_n v_n,$$

where $\sum_{i=0}^n r_i = 1$.
e) Every point in the convex hull $\operatorname{conv}(\{v_0, \ldots, v_n\})$ *has a unique representation in the form*

$$r_0 v_0 + \ldots + r_n v_n,$$

where $\sum_{i=0}^n r_i = 1$ *and* $r_i \geq 0$ *for all* i.

Proof. a)$\Longleftrightarrow$ b) Condition a) means that

$$r_1(v_1 - v_0) + r_2(v_2 - v_0) + \ldots r_n(v_n - v_0) = 0$$

if and only if $r_1 = \ldots = r_n = 0$. This is the same as

$$r_1 v_1 + r_2 v_2 + \ldots r_n v_n - (r_1 + \ldots + r_n)v_0 = 0$$

if and only if $r_1 = \ldots = r_n = 0$. Letting $r_0 = -(r_1 + \ldots + r_n)$ and noticing that then $r_0 + r_1 + \ldots + r_n = 0$, we see that this is the same as condition b).

b)$\Longleftrightarrow$ c)Clear, since conditions of c) are equivalent to

$$\sum_{i=0}^n (r_i - r'_i)v_i = 0 \text{ and } \sum_{i=0}^n (r_i - r'_i) = 0.$$

c)$\Rightarrow$ d) Condition d) is a special case of condition c) for $\sum_{i=0}^n r_i = 1$.
d) $\Rightarrow$ e) Clear.
e)$\Rightarrow$ b) Suppose

$$\sum_{i=0}^n r_i v_i = 0 \text{ and } \sum_{i=0}^n r_i = 0.$$

We need to prove that $r_i = 0$ for all $i = 0, \ldots, n$. Suppose they are not. Then there are indices for which $r_i > 0$ and indices for which $r_i < 0$ (since $\sum_{i=0}^n r_i = 0$). We may assume that $r_0, \ldots, r_k \geq 0$, $r_{k+1}, \ldots, r_n < 0$ for $0 \leq k < n$. Define $r = r_1 + \ldots + r_k = -r_{k+1} - \ldots - r_n > 0$. We have

$$r_0 v_0 + \ldots + r_k v_k = (-r_{k+1})v_{k+1} + \ldots + (-r_n)v_n \text{ hence}$$

$$\frac{r_0}{r} v_0 + \ldots + \frac{r_k}{r} v_k = \frac{-r_{k+1}}{r} v_{k+1} + \ldots + \frac{-r_n}{r} v_n.$$

Both sides are convex combinations of points $v_0, \ldots, v_n$. Thus we obtain different convex combinations for the same point, which is a contradiction with e). $\square$

Definition 1.1.4. *If the set* $\{v_0, \ldots, v_n\}$ *satisfies one (hence all) conditions of the previous lemma it is called* **affinely independent**.

Definition 1.1.5. *A convex hull of the affinely independent finite set* $\{v_0, \ldots, v_n\}$ *is called the* n**-dimensional simplex** *with vertices* $v_0, \ldots, v_n$.

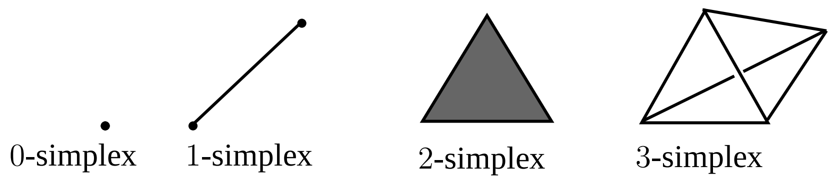
A simplex is usually denoted by the symbol σ. The set of its vertices $\{v_0, \ldots, v_n\}$ is uniquely determined by the set σ itself (Exercise 1.4), so we can talk about the vertices and the dimension of the simplex unambiguously. Conversely the set of vertices $\{v_0, \ldots, v_n\}$ defines the simplex uniquely. The fact that $\{v_0, \ldots, v_n\}$ are vertices of a simplex σ is also expressed by saying that the points $v_0, \ldots, v_n$ **span** a simplex σ.

Notice that an n-dimensional simplex has $n+1$ vertices. By the Lemma 1.1.3 every point x of the simplex σ with vertices $\{v_0, \ldots, v_n\}$ can be written in the form

$$x = r_0 v_0 + r_1 v_1 + \ldots + r_n v_n,$$

where $r_i \geq 0$ and $\sum_{i=0}^{n} r_i = 1$ in a unique way.
If above $r_i > 0$ for all $i \in \{0, \ldots, n\}$, we say that x is **an interior point** of the simplex. The set of all interior points is called **the interior of the simplex** and will be denoted by $\operatorname{int} \sigma$. The points which are not interior are called **boundary points**. The set of boundary points is called **the boundary of the simplex**, denoted by $\partial \sigma$.

0-simplex is a point, 1-simplex is a closed interval, 2-simplex is a triangle, 3-simplex is a tetrahedron.



Examples 1.1.6. *The canonical example of the n-dimensional simplex is the set*

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \textit{ for all } i, \sum_{i=1}^{n} x_i \leq 1\}.$$

To see that this is indeed a simplex, consider the set $\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$. *Lemma 1.1.3 easily implies that this set is affinely independent. It is straightforward to verify that* Δ_n *is a convex hull of these points. For the notacional convenience we will denote* $0 = e_0$. *Hence the simplex* Δ_n *is spanned by the* *points* $e_0, \ldots, e_n$.

Another, even simpler example is

$$\Delta'_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} x_i = 1\}.$$

Here the set of vertices is exactly the standard basis of $\mathbb{R}^{n+1}$ *- the set* $\{e_0, \ldots, e_n\}$. *It is easy to see that both* Δ_n *and* Δ'_n *are compact Hausdorff spaces (Exercise 1.6).*

An affine mapping $f\colon \sigma \to \sigma'$, where σ and σ' are both simplices is called **simplicial** if for every vertex v_i of σ the image $f(v_i)$ is a vertex of σ'.

Lemma 1.1.7. *Suppose* $\sigma \subset V$ *is an* n*-dimensional simplex with vertices* $\{v_0, \ldots, v_n\}$, $C \subset V'$ *is a convex set and* $w_0, \ldots, w_n \in C$ *are arbitrary points. Then there exists a unique affine mapping* $f\colon \sigma \to C$ *such that* $f(v_i) = w_i$ *for all* $i = 0, \ldots, n$.

Proof. Uniqueness is clear - since every point $x \in \sigma$ has a unique representation

$$x = r_0 v_0 + r_1 v_1 + \ldots + r_n v_n$$

as a convex combination, we must have (exercise 1.3c)

$$f(x) = r_0 f(v_0) + \ldots r_n f(v_n) = r_0 w_0 + \ldots + r_n w_n.$$

Conversely this formula defines a well-defined mapping (since C is convex) and the verification that it is affine is left to the reader. □

If V is a finite-dimensional vector space every simplex of V has a natural (relative) topology as a subspace. In case V is not finite-dimensional it does not have a priori any natural topology. However, the previous lemma implies that for every n-simplex σ in V (finite-dimensional or not) there exists a unique simplicial mapping $f\colon \Delta_n \to \sigma$ defined by $f(e_i) = v_i$ for all $i = 0, \ldots, n$ (where $v_0, \ldots, v_n$ are vertices of σ listed in some order). We have

$$f(x_1, \ldots, x_n) = x_0 v_0 + x_1 v_1 + \ldots + x_n v_n,$$

where $x_0 = 1 - x_1 - \ldots - x_n$. Since both sets of vertices are affinely independent, it follows that f is a bijection. In case V is finite-dimensional, i.e. has a natural topology, f is clearly continuous, since operations of addition and multiplication by real numbers are continuous in V. Since σ is Hausdorff and Δ_n is compact, it follows that f is then a homeomorphism. Thus in the general case we can define a topology on Δ by requiring that f is a homeomorphism. This topology will be referred to as **the Euclidean topology on** σ. Notice that the homeomorphism f above depends on the choice of the order of vertices. However, the next Proposition shows that the Euclidean topology does not depend on the order of vertices.

Proposition 1.1.8. *Suppose $\sigma \subset V$ is an n-dimensional simplex with vertices $\{v_0, \dots, v_n\}$. Then σ equipped with its Euclidean topology is a compact Hausdorff space.*
If $C \subset V'$ is a convex set, where V' is finite-dimensional, then any affine mapping $f: \sigma \to C$ is continuous.

In particular if σ' is another n-simplex with vertices $\{v'_0, \dots, v'_n\}$, then there exists a unique simplicial mapping $f: \sigma \to \sigma'$ for which $f(v_i) = v'_i$ and this mapping is necessarily a homeomorphism. Hence two n-dimensional simplices are homeomorphic as topological spaces and in particular Euclidean topology does not depend on the chosen order of vertices.

Proof. Since σ equipped with the Euclidean topology is homeomorphic to Δ_n, it is a compact Hausdorff space (Exercise 1.6). Via this homeomorphism affine mapping $f: \Delta_n \to C$ is defined by the formula

$$f(x_1, \dots, x_n) = x_1 w_1 + \dots + x_n w_n + (1 - x_1 - \dots - x_n) w_0,$$

where $w_i = f(v_i) \in C$. This is certainly continuous with respect to the Euclidean topology in V'.

In particular, every simplicial mapping between two simplices is continuous, hence every simplicial bijection between two n-simplices is a homeorphism (notice that the inverse of a simplicial bijection is also a simplicial mapping). In particular if we consider σ with two different orders of vertices, the identity mapping $\mathrm{id}: \sigma \to \sigma$ is a homemorphism. Hence the Euclidean topology does not depend on the order of vertices. $\square$

It is easy to see that an n-simplex is homeomorphic to the closed n-dimensional ball $\overline{B}^n$, in fact this follows also from the more general Proposition 1.1.10 we prove below. One of the questions that originally led to the development of algebraic methods in topology was the question whether $\overline{B}^n$ and $\overline{B}^m$ are homeomorphic for $n \neq m$. Later we will prove that the answer is the intuitively expected "no".

Lemma 1.1.3 easily implies that any subset of an affinely independent set is also affinely independent. Hence if σ is a simplex with vertices $\{v_0, \ldots, v_n\}$, any subset of vertices $\{v_{i_0}, \ldots, v_{i_k}\}$ spans a simplex σ', which is clearly a subset of σ. Such a simplex is called **a face of** σ and we also denote this as $\sigma' < \sigma$. In particular $n-1$-dimensional faces of σ are simplices with vertices $\{v_0, \ldots, \widehat{v_i}, \ldots, v_n\}$, where $\widehat{v_i}$ symbolises that the element v_i is omitted.

As we already observed above, a simplex defines the set of its vertices $\{v_0, \ldots, v_n\}$ uniquely, but of course the ordering in which vertices are listed can be arbitrary - any permutation defines the same simplex. For technical reasons that will become apparent later it is in some contexts convenient to fix the ordering of vertices. For example we would like to assign to every $n-1$-dimensional face $\{v_0, \ldots, \widehat{v_i}, \ldots, v_n\}$ the number i (the index, which the omitted vertex has) and call it the ith face of the simplex. This is not possible if we treat the set of vertices merely as a set.

Definition 1.1.9. An ordered simplex *is a pair* $(\sigma, \leq)$*, where* σ *is a simplex and* $\leq$ *is a linear order defined on a set of its vertices.*

Since simplex is completely determined by the set of its vertices, ordered n-simplex can be identified with $(n+1)$-tuple $(v_0, \ldots, v_n) \in V^{n+1}$, where $v_0 < v_1 < \ldots < v_n$.
An order of vertices of an ordered simplex σ defines by restriction an order on any face of σ. Hence we can (and will) consider any face of an ordered simplex as an ordered simplex as well. In an ordered n-simplex every $(n-1)$-dimensional face can be given an index - we call the face with vertices $(v_0, \ldots, \widehat{v_i}, \ldots, v_n)$ the i**th face** of the ordered simplex σ. The expression "**face opposite to the vertex** v_i" is also used.

We conlude this section with an interesting topological fact about bounded convex sets.

Proposition 1.1.10. *Suppose* $C \subset V$ *is a bounded closed convex subset, where* V *is an* n*-dimensional vector space. Then* C *is homeomorphic to the closed ball* $\overline{B}^k$ *for* $-1 \leq k \leq n$ *(where* $k = \dim \operatorname{aff}(C)$*) via a homeomorphism which maps the boundary of* C *(with respect to* $\operatorname{aff} C$*) to* $\partial B^k = S^{k-1}$ *and the interior of* C *(with respect to* $\operatorname{aff} C$*) to the open ball* B^k*.*

Proof. If C is empty, claim is trivially true for $k=-1$. Hence we may assume that $C \neq \emptyset$.

According to Exercise 1.5 in this case C has an interior point with respect to $\operatorname{aff}(C)$. By translating we may assume that C contains 0 as the interior point with respect to $\mathbb{R}^k$, where $k = \dim \operatorname{aff}(C)$. Hence it is enough to prove that if 0 is an interior point of C, then C is homeomorphic to $\overline{B}^n$ via the homeomorphism that maps interior to interior and boundary to boundary.
First we define a mapping $f\colon \partial C \to S^{n-1}$ by $f(x) = x/|x|$. This is well-defined and continuous, since 0 is not a boundary point of C. Since C is bounded, for every $y \in S^{n-1}$ the connected set (half-line)

$$L_y = \{ty \mid t \geq 0\} \subset \mathbb{R}^n$$

interects both C (in 0) and its complement. Hence it also intersects the boundary of C. This implies that f is surjective. To show that f is actually bijective it is enough to prove that the half-line L_y intersects ∂C at precisely one point. Let

$$t_0 = \sup\{t \mid ty \in C\}.$$

Then $t_0 > 0$, since 0 is an interior point of C, and t_0 is certainly a finite real number, since C is bounded. Suppose $x = t_0 y$, then $x \in \partial C$. It is enough to show that $ty \notin \partial C$ for $t < t_0$ and $ty \notin C$ for $t > t_0$. Define

$$W = \bigcup_{0 \leq t < 1} (1-t)U + tx,$$

where U is an open neighbourhood of 0 contained in C. Then W is open in V and is a subset of C, since C is convex. Moreover the half-open interval $[0, x[$ is contained in W. Hence $tx \in \operatorname{int} C$ and consequently not a boundary point for all $t < 1$. This proves that $t_0 y \notin \partial C$ for $t_0 < t$.

Next consider the case $t > t_0$. By the definition of supremum there exists $t' \in]t_0, t[$ such that $x' = t'y \notin C$. Suppose $ty \in C$. Then by convexity $t'y \in C$, since it belongs to the closed interval $[x, ty]$. This is a contradiction, so $ty \notin C$.

We have shown that f is a continuous bijection. Since both ∂C and S^{n-1} are compact and Hausdorff spaces, f is a homeomorphism.
The considerations above also show that $ty \in \operatorname{int} C$ for all $t \in [0, t_0[$.

To complete the proof we must extend this homeomorphism to the interior of C. This is now just scaling - one maps the interval $[0, x]$ to the interval $[0, x/|x|]$ in a linear manner for every boundary point x. It is actually easier to do it the other way around - define $G\colon \overline{B}^n \to C$ by

$$G(t) = |t| \cdot f^{-1}\Big(\frac{t}{|t|}\Big) \text{ if } t \neq 0$$

and $G(0) = 0$. We leave the verification that G is a homeomorphism to the reader as an exercise(1.7). $\square$

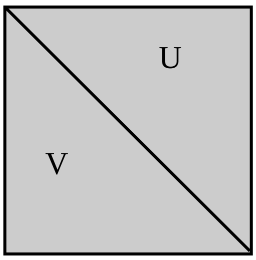
The last part of the claim of the previous proposition is a bit redundant - every homeomorphism $f\colon \overline{B}^k \to \overline{B}^k$ must map interior to interior and boundary to boundary. This fact (related to the famous "Invariance of Domain" theorem) seems obvious, but is not easy to prove, just as the fact that Euclidean spaces of different dimension are non-homeomorphic. Both are consequences of the invariance of domain theorem, which we will prove in this course.

Since any n-simplex σ is a closed convex set that has interior points (namely the interior of a simplex, as defined above) with respect to the n-dimensional affine set $\operatorname{aff}\sigma$ we obtain the following result.

Corollary 1.1.11. *Suppose σ is an n-simplex. Then there exists a homeomorphism $\sigma \to \overline{B}^n$ that maps $\operatorname{int}\sigma$ to B^n and $\partial\sigma$ to S^{n-1}.*

1.2 Simplicial complexes

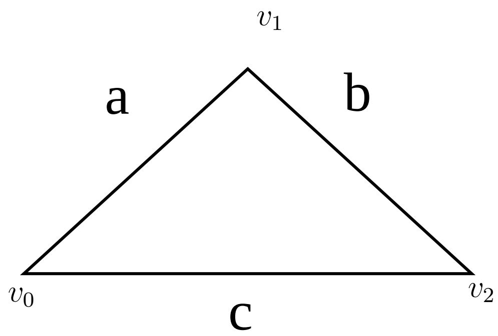
One of the reasons simplices were invented and are useful, is that many familiar spaces/geometric figures, although not simplices themselves, can be built out of simplices by "gluing " them together in a "regular" manner. For example a square is not a simplex (can you come up with an easy argument why not?) but if you "cut" it along the diagonal you will easily see that it is obtained from two triangles i.e. 2-simplices which have a common side - namely the diagonal itself. This is illustrated in the picture below - U and V are right-angled triangles with one common side.



Of course a square is homeomorphic to a simplex anyway, since it is convex, so there may seem little point in representing it in this way from the topologist point of view, although we will see a bit later, when we talk about Δ-complexes, that a little generalization of this idea might be a fruitful way of representing some less "simple" spaces, such as the Klein's bottle or the

Mobius band.

As a little more interesting example consider the boundary of a 2-simplex i.e. a triangle with vertices $\{v_0, v_1, v_2\}$ - see the picture below. Now this boundary is not homeomorphic to a simplex (although we can't quite prove it yet, we will later and it seems very believable anyway). But still, just as with the square above, one can think of this boundary as the union of three 1-simplices (a, b and c in the picture) such that two of them always intersect at a vertex (and every vertex is a common face of precisely two 1-simplices).



The spaces obtained in this way are called **polyhedrons**. Since every simplex is determined by a finite set of its vertices, in this fashion we obtain a purely combinatorical " discrete " representation of a topological space in question - sort of like a " skeleton " of the space. Let us now switch to formal definitions.

Definition 1.2.1. *Suppose V is a vector space (not necessarily finite-dimensional). A collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is called* **a (geometric) simplicial complex** *if the following conditions are satisfied:*
1) For every simplex σ in K its every face also belongs to K.
2) For every pair σ, σ' of simplices in K their intersection is either empty or a common face of σ and σ'.

A useful alternative definition of a simplicial complex is formulated in the following lemma.

Lemma 1.2.2. *Suppose V is a vector space. A collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is a simplicial complex if and only if*
1) For every simplex σ in K its every face also belongs to K.
2') For every $x \in \bigcup_{i \in I} \sigma_i$ there is a unique $i \in I$ such that x is an interior point of the simplex σ_i.

Proof. Exercise 1.8 □

A subset L of K which is a simplicial complex on its own is called **a simplicial subcomplex** of K. Notice that for any subset of K the condition 2)' in the Lemma above is satisfied automatically, so for $L \subset K$ to be a subcomplex it is enough that it satisfies condition 1) i.e. is closed under faces of its simplices.
If L is a subcomplex of K we call the pair (K, L) **a pair of simplicial complexes**.

The underlying set

$$|K| = \bigcup_{i \in I} \sigma_i \subset V$$

is called **a polyhedron of the complex** K. If V is finite-dimensional this set, of course, has a natural Euclidean topology inherited from the standard topology of V, but this is not necessarily the topology we are interested in. If V is not finite-dimensional we don't have any canonical topology in V, that might define a relative topology on this polyhedron, but in fact we don't need one. There is a standard way to define a topology on any polyhedron.

We start off by noticing that our polyhedron is a union of simplices anyway, and every simplex has its standard Euclidean topology. All we need to do is to "glue together" this topologies to obtain a topology on $|K|$. For this we need the following general topological result.

Proposition 1.2.3. *Suppose X is a set and $(X_i)_{i \in I}$ is a collection of its subsets and assume every subset X_i is given a topology τ_i. Suppose also that*
1) For all $i, j \in I$ the relative topologies induced on $X_i \cap X_j$ by τ_i and τ_j coincide.
2) For all $i, j \in I$ the set $X_i \cap X_j$ is closed in X_i with respect to the relative topology induced by τ_i.
Then there exists a unique topology τ on X such that a subset $A \subset X$ is open (closed) in (X, τ) if and only if $A \cap X_i$ is open (closed) in (X_i, τ_i) for all $i \in I$. Moreover the relative topology induced by τ on X_i coincides with τ_i and X_i is closed in X for every $i \in I$.

Proof. A topology τ described by the condition $A \subset X$ is open(closed) in (X, τ) if and only if $A \cap X_i$ is open(closed) in (X_i, τ_i) always exists and unique - in fact it is the topology induced by the inclusions $X_i \hookrightarrow X$, $i \in I$.
All we need to prove is that this topology satisfies the other conditions. First of all by condition 2) we see that X_i is closed in X by the very definition of induced topology τ.

Suppose $A \subset X_i$ is closed with respect to τ_i. For every $j \in J$ the set $A \cap X_j$ is closed in $X_i \cap X_j$ with respect to the relative topology induced by τ_i, hence also closed in $X_i \cap X_j$ with respect to the relative topology induced by τ_j, by 1). Since $X_i \cap X_j$ is closed in (X_j, τ_j), it follows that $A \cap X_j$ is closed in (X_j, τ_j). Hence by the defintiion A is closed in (X, τ). In particular A is closed in the relative topology of X_i induced by τ.

Conversely suppose $A \subset X_i$ is closed in X_i with respect to the relative topology induced by τ. Since X_i is closed in (X, τ) it follows that A is also closed in (X, τ). By the definition of τ this means that in particular $A = A \cap X_i$ is closed in (X, τ_i). $\square$

The topology on the set X is called **coherent** with a family $(X_i)_{i \in I}$ of subsets of X if a subset $A \subset X$ is open (closed) in X if and only if $A \cap X_i$ is open (closed) in X_i for every $i \in I$(with respect to relative topology).

Returning to our polyhedron, it is easy to see that the collection of simplices $\sigma \in K$ equipped with their Euclidean topologies satisfies the conditions of Proposition 1.2.3 for the set $|K|$. Namely the intersection of two simplices σ, σ' is either empty or a common face. Clearly the Euclidean topologies induce on the intersection $\sigma \cap \sigma'$ the Euclidean topology of this simplex, which is closed in both σ and σ'. Hence Proposition 1.2.3 implies that there is a unique topology on $|K|$ which is coherent with Euclidean topologies of all simplices $\sigma \in K$ and induces the Euclidean topology on every simplex. Moreover every simplex is then closed in $|K|$.
From now on when we talk about a polyhedron of a simplicial complex, we assume it is equipped with this topology, which we will call **weak topology**. Notice that even if V is finite-dimensional, in which case $|K|$ has a relative topology as its subset, this topology is NOT necessarily the same as the weak topology defined above.

Example 1.2.4. *Consider the subset* $\{0\} \cup \{1/n \mid n \in \mathbb{N}_+\} \subset \mathbb{R}$. *We can think of this set as a simplicial complex consisting of* 0*-simplices, which are just points. The weak topology on this set is simply the discrete topology. However in the relative topology as a subset of* $\mathbb{R}$ *this set is not descrete, since* 0 *is a limit point of the sequence* $\{1/n \mid n \in \mathbb{N}_+\}$.
As a more extreme, not interesting trivial example one could even take as a set of 0*-simplexes the whole vector space* $\mathbb{R}^n$, *thus obtaining* 0*-dimensional polyhedron with descrete topology, whose underlying set is* $\mathbb{R}^n$.

We will, however, be mainly interested in finite simplicial complexes, for which no such problem can arise.

Proposition 1.2.5. *Suppose K is a simplicial complex in a vector space V. Then $|K|$ is compact with respect to the weak topology if and only if K is finite.*
If K is finite and V is finite dimensional, then the weak topology in $|K|$ coincides with the relative topology of the subspace of V.

Proof. Every simplex is compact. Hence a finite simplicial complex is compact, as a finite union of compact spaces.
Conversely suppose K is not finite. For every $\sigma \in K$ choose a point $x_\sigma \in \operatorname{int} \sigma$. By Lemma 1.2.2 $x_\sigma \neq x'_\sigma$ if $\sigma \neq \sigma'$. Hence the set

$$C = \{x_\sigma \mid \sigma \in K\}$$

is infinite. Let A be an arbitrary subset of C. Now every simplex of K intersects A in a finite set (at most one point of C is in the interior of every face of the simplex), which is certainly closed in this simplex. Hence by the definition of the weak topology A is closed in $|K|$. Hence in particular C is closed in $|K|$ and its every subset is closed, so it has a discrete topology. If $|K|$ was compact, also C would be compact. But a compact discrete space is always finite. Hence we obtain the contradiction.

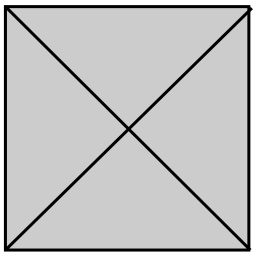
For the second claim it is enough to notice the following. Suppose X is a topological space which is a finite union of **closed** subsets $A_1, \ldots, A_n$. Then the topology of X is coherent with the family (A_i). This is proved in Topology II (or prove it yourself, it is very easy). □

If L is a subcomplex of K, the space $|L|$ has the Euclidean topology, as a polyhedron of a simplicial complex on its own, and the relative topology induced from the Euclidean topology of K. Both topologies coincide. Moreover $|L|$ is closed in $|K|$ (Exercise 2.10).

A triangulation of a topological space X is a pair (K, f) where K is a simplicial complex and $f\colon X \to |K|$ is a homeomorphism. A space that has a triangulation is called **a polyhedron**. **A pair** (X, Y) **of topological spaces** (i.e. Y is a subspace of X) is called **a polyhedron pair** if there exists a triangulation $f\colon X \to |K|$ and a subcomplex L of K such that $f^{-1}|L| = Y$.

Examples 1.2.6. *1 As already noted in the beginning of this section, a square can be represented as a polyhedron of a simplicial complex that* *contains* 2 2*-simplices with one common side and all their faces. For instance if the vertices of the square are the points* $(0,0), (0,1), (1,0), (1,1)$ *in the plane* $\mathbb{R}^2$*, then as a suitable simplicial complex we can take a complex consisting of* 2*-simplices* $\{(0,0),(0,1),(1,0)\}$, $\{(0,1),(1,0),(1,1)\}$, 1*-simplices* $\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$, $\{(1,0),(0,1)\}$, $\{(1,1),(0,1)\}$, $\{(1,1),(1,0)\}$ *and* 0*-simplices* $\{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}$.

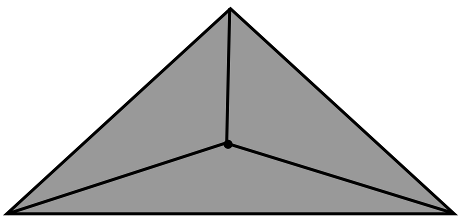
Of course there are other ways to triangulate a square - we could use another diagonal to subdivide it into 2 *triangles or both diagonals to subdivide it into a polyhedron of a simplicial complex with* 4 2*-simplices (see the picture).*



So we see that there are many ways to triangulate a given polyhedron.

2) *Suppose* σ *is* n*-dimensional simplex in a vector space* V*. Then its faces define a simplicial complex, which we also denote by* $K(\sigma)$ *and its proper faces, i.e. all of its faces except* σ *itself, define a simplicial complex which is denoted* $K(\partial\sigma)$*. Clearly* $K(\partial\sigma)$ *is a subcomplex of* $K(\sigma)$*. Of course* $|K(\sigma)| = \sigma$ *and* $|K(\partial\sigma)| = \partial\sigma$*. In particular we see that* S^{n-1} *is a polyhedron and the pair* $(\overline{B}^n, S^{n-1})$ *is a polyhedron pair (by Proposition 1.1.10).*

3) *Again we can subdivide* σ *to obtain the other, less trivial representations of* σ *as a polyhedron - for one example see the picture below.*



Here K *has three* 2*-simplices and their faces. The simplicial complex* *is different from $K(\sigma)$ but the underlying polyhedron is the same.*

4) *Suppose K is a simplicial complex and $n \in \mathbb{N}$. The collection of all simplices of K with dimension $\leq n$ is clearly a subcomplex of K, which we denote K^n and call* **the nth skeleton** *of K.*
   The elements of $|K^0|$ are called **the vertices** *of the simplicial complex K.*

   The simplicial complex K is called **finite-dimensional** *if $K = K^n$ for some $n \in \mathbb{N}$. The smallest n that satisfies this condition is then called* **the dimension** *of K. If K is not finite-dimensional, we say that it is* **infinite dimensional**.

   If K is 0-dimensional, the weak topology of $|K|$ is the discrete topology.

According to Lemma 1.2.2 every point $x \in |K|$ is an interior point of the unique simplex $\sigma \in K$. This simplex is called **the carrier** of x and is denoted by $\mathrm{car}(x)$.
The star of x is defined to be the set

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid x \in \sigma\}.$$

Lemma 1.2.7. *Suppose $x \in |K|$ and the set of vertices of $\mathrm{car}(x)$ is $\{v_0, \ldots, v_n\}$. Then*

a) $\mathrm{St}(x)$ *is an open neighbourhood of x in $|K|$.*

b)

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid \mathrm{car}(x) < \sigma\} = \bigcup\{\mathrm{int}\,\sigma \mid v_0, \ldots, v_n \text{ are vertices of } \sigma\}.$$

.

c)

$$\mathrm{St}(x) = \bigcap_{i=0}^{n} \mathrm{St}(v_i).$$

Proof. Exercise 2.11 □

One of the reasons homology theory works as well as it does is that simplicial complexes can be subdivided into simplicial complexes with " arbitrary small " simplices.

Definition 1.2.8. *A simplicial complex K' is* **a subdivision** *of a simplicial complex K if*
1) Every simplex of K' is a subset of some simplex of K.
2) Every simplex of K is a finite union of some simplices of K'.

If K' is a subdivision of K it follows straight from the definition that $|K'| = |K|$. Moreover the Euclidean topology induced by K' and K on the set $|K| = |K'|$ coinside (Exercise 2.12).

The important canonical subdivision of a given simplicial complex is the so-called **barycentric division**. It is constructed as follows.
Suppose K is a simplicial complex. Let $\sigma \in K$ be an n-simplex with vertices $\{v_0, \ldots, v_n\}$. The point

$$b = b(\sigma) = \frac{1}{n+1}(v_0 + v_1 + \ldots + v_n) \in \sigma$$

is called **a barycentre** of the simplex σ.

Lemma 1.2.9. *Suppose $\sigma_0 < \sigma_1 < \ldots < \sigma_n$ is a linearly ordered finite chain of simplices (where $\sigma_n \in K$), where σ_i is a face of σ_j for $i < j$.*
Then the set of barycentres $\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\}$ is affinely independent, hence defines an n-simplex, which is a subset of σ_n.

Proof. We prove the claim by induction on n. For $n = 0$ the claim is clear. Suppose

$$r_0 b(\sigma_0) + r_1 b(\sigma_1) + \ldots + r_n b(\sigma_n) = 0,$$

where $r_0 + \ldots + r_n = 0$. It is enough to prove that $r_n = 0$.

Let $\{v_0, \ldots, v_m\}$ be the set of vertices of σ_n. We may assume that σ_{n-1} (hence σ_i for all $i < n$) is a face of a simplex spanned by $\{v_0, \ldots, v_{m-1}\}$ Now every barycentre $b(\sigma_i)$ can be written as a convex combination

$$b(\sigma_i) = a_0^i v_0 + a_1^i v_1 + a_m^i v_m,$$

where $\sum_{j=0}^m a_j^i = 1$ and $a_m^i = 0$ for $i < n$. Substituting this expression in the equation above gives us an equation

$$r'_0 v_0 + \ldots + r'_m v_m = 0, \text{ where}$$

$$r'_j = \sum_{i=0}^n r_i a_j^i.$$

Simple computation implies that

$$\sum_{j=0}^{m} r'_j = \sum_{j=0}^{m}\sum_{i=0}^{n} r_i a^i_j = \sum_{i=0}^{n} r_i \sum_{j=0}^{m} a^i_j = \sum_{i=0}^{n} r_i = 0.$$

Since $\{v_0, \ldots, v_m\}$ is affinely independent, this implies that $r'_j = 0$ for all $j = 0, \ldots, m$, in particular $r'_m = 0$. But on the other hand

$$r'_m = \frac{r_n}{n+1}.$$

Hence $r_n = 0$ and we are done. □

Proposition 1.2.10. *Suppose K is a simplicial complex. Define K' as the collection of simplices* $\operatorname{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\})$, *where $\sigma_0 < \sigma_1 < \ldots < \sigma_n \in K$. Then K' is a simplicial complex and it is a subdivision of K.*

Proof. Clearly

$$\operatorname{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\}) \subset \sigma_n.$$

Let $\sigma \in K$ be arbitrary n-simplex with vertices $\{v_0, \ldots, v_n\}$. Let $x \in \sigma$. By the proof of Proposition 1.1.10 we know that x is either a barycentre or there is a unique $y \in \partial\sigma$ such that x is on the interval between barycentre (interior zero point in the proof of 1.1.10) and y. In other words there are unique $y \in \partial\sigma$ and $r \in]0, 1]$ such that

$$x = rb(\sigma) + (1 - r)y.$$

Now y is a point of a proper face $\sigma' < \sigma$. Assume, by induction (on the dimension n), that every face σ' of σ can be written as a finite union of all the possible simplices of K' of the form

$$\operatorname{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\}),$$

where σ_n is a face of σ'. By the equation

$$x = rb(\sigma) + (1 - r)y$$

we get the similar result for σ. The only exception - the barycentre itself- corresponds then to the 0-simplex $\{b(\sigma)\}$.
Since the claim is clear for $n = 0$ (every vertex of K is a vertex of K'), we obtain by induction that every simplex of K is a finite union of the simplices of K'.
It remains to show that K' is indeed a simplicial complex. Since it is obviously closed under faces, lemma 1.2.2 implies that it is enough to prove that every

point of $|K'| = |K|$ is an interior point for the **unique** simplex $\sigma' \in K'$. Hence suppose $x \in |K|$ is an interior point of a simplex

$$\mathrm{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\})$$

i.e.

$$x = \sum_{i=0}^{n} a_i b(\sigma_i),$$

where $\sum_{i=0}^{n} a_i = 1$, $a_i > 0$. This implies in particular that $x \in \mathrm{int}\, \sigma_n$, so σ_n is uniquely determined by x. Letting $a = \sum_{i=0}^{n-1} a_i = 1 - a_n$ we can write this as

$$x = (1 - a_n)y + a_n b(\sigma_n),$$

where y is a boundary point of σ_n, belonging to σ_{n-1}. On the other hand we already know that this expression is unique, if $x \neq b(\sigma_n)$. By induction we may assume that y is an interior point of the unique simplex in K', which now will have to be

$$\mathrm{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_{n-1})\}).$$

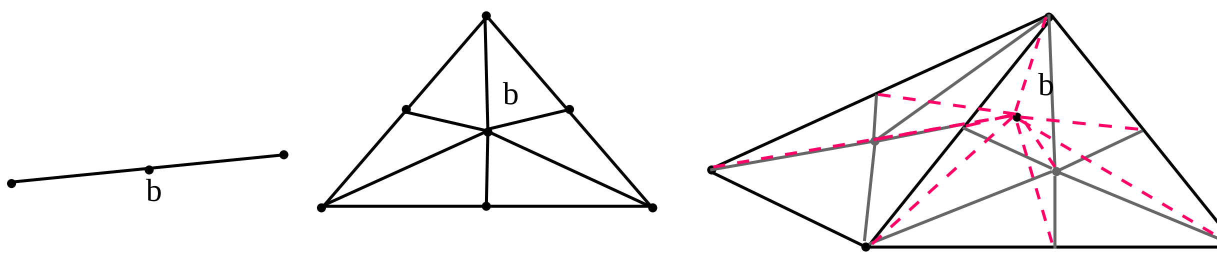
This implies that

$$\mathrm{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\})$$

is uniquely determined by x, i.e. is the only possible simplex of K', whose interior contains x.
In case x IS a barycentre, y above is not unique, but $a_n = 1$, so this shows that x can only belong to the interior of the 0-simplex $\{b(\sigma)\}$.
The proposition is proved. □

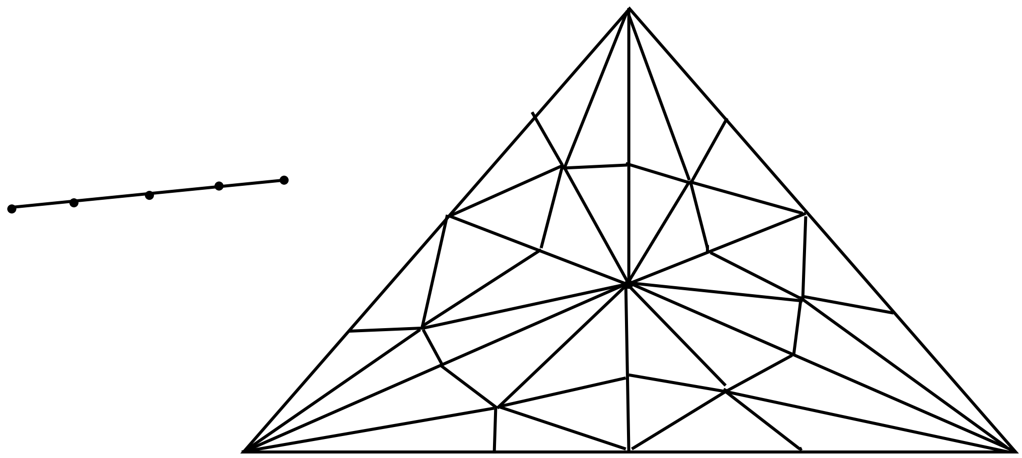
Definition 1.2.11. *Suppose K is a simplicial complex. A simplicial complex K' defined in the previous proposition is called* **the first barycentric division of K**.

The following picture illustrates the barycentric subdivision of 1 and 2 simplices as well as the part of the barycentric subdivision of a 3 simplex, where only subdivision of two front faces and lines from the barycentre to the visible vertices are shown. The barycentre of the whole simplex is denoted b.



The construction can be iterated - suppose K' is a first barycentric division of K and let K'' be a first barycentric division of K'. Then K'' is called **the second barycentric division of** K.
This can be continued by induction. Suppose $(n-1)$:th barycentric division $K^{(n-1)}$ of K is defined. We define **the n-th barycentric division** $K^{(n)}$ to be the first barycentric division of $K^{(n-1)}$. Hence the notation $K^{(1)}$ will be used for the first barycentric division. For convinience we also define $K^{(0)} = K$.

The following picture illustrates the second barycentric subdivision of a 1-simplex and a 2-simplex. You can see, how simplices are getting smaller with each subdivision.



Barycentric divisions are most useful for finite simplicial complexes. To formulate and prove next results we need the concept of the diameter of a simplex, hence the concept of the linear metric on a simplex. Of course every finite simplicial complex can be considered a simplicial complex in a finite-dimensional space V, which can be identified with $\mathbb{R}^m$, and hence given a linear metric. This metric will of course depend on the chosen identification $V = \mathbb{R}^m$. For our purposes it is enough to consider simplicial complex which already are complexes in some $\mathbb{R}^m$, hence have a natural metric.

Lemma 1.2.12. *Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{v_0, \ldots, v_n\}$. Then*

$$\operatorname{diam} \sigma = \max\{|v_i - v_j|\},$$

where $|\cdot|$ is a standard norm on $\mathbb{R}^m$.

Proof. Exercise 1.13) $\square$

Lemma 1.2.13. *Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in a first barycentric division K', with vertices $\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\}$, where $\sigma_0 < \ldots < \sigma_n = \sigma \in K$. Then*

$$\operatorname{diam} \sigma' \leq \frac{n}{n+1} \operatorname{diam} \sigma$$

Proof. Exercise 1.13b) □

For a finite simplicial complex $K \subset \mathbb{R}^m$ we define its mesh by

$$\operatorname{mesh} K = \max\{\operatorname{diam} \sigma \mid \sigma \in K\}.$$

Corollary 1.2.14. *Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that*

$$\operatorname{mesh} K^{(n)} < \varepsilon.$$

Proof. Let m be a maximal dimension of a simplex in K. Since $\frac{k}{k+1} < 1$ for every $k \leq m$, there exists $n \in \mathbb{N}$ such that

$$\left(\frac{k}{k+1}\right)^n \operatorname{mesh} K < \varepsilon.$$

Iterration of the result of the previous lemma shows that then

$$\operatorname{mesh} K^n < \varepsilon.$$

□

Now we can finally prove the important result that shows that a compact polyhedron has " arbitrary fine " triangulations.
Suppose $\mathcal{U}$ is **an open covering** of $|K|$. Recall that this means that every $U \in \mathcal{U}$ is an open set of $|K|$ and

$$|K| = \bigcup \{U \in \mathcal{U}\}.$$

We say that K is **finer** than $\mathcal{U}$ if for every vertex $v \in K$ there is $U \in \mathcal{U}$ such that $\mathrm{St}(v) \subset U$. In other words the open covering

$$\{St(v) \mid v \text{ is a vertex of } K\}$$

is finer (is a refinement) than the covering $\mathcal{U}$.
More generally a triangulation (K, f) of a polyhedron X is said to be **finer** than the open covering $\mathcal{U}$ of X, if the covering

$$\{f^{-1}St(v) \mid v \text{ is a vertex of } K\}$$

is a refinement of $\mathcal{U}$.

Proposition 1.2.15. *Suppose K is a finite simplicial complex and $\mathcal{U}$ is an open covering of $|K|$. Then there exists $n \in \mathbb{N}$ such that n-th barycentric division $K^{(n)}$ is finer then $\mathcal{U}$.*

Proof. Since K is finite, the affine subspace that vertices of its simplices generate is final-dimensional, so we might as well assume that K is a subset of final-dimensional vector space V. By inducing metric on V via some linear homeomorphism $V \cong \mathbb{R}^m$ we might actually assume that K is a simplicial complex in $\mathbb{R}^m$ for some $m \in \mathbb{N}$.
Since $|K|$ is compact (lemma 1.2.5), there is $\varepsilon > 0$ such that any subset $A \subset |K|$ with $\operatorname{diam} A < \varepsilon$ is contained in some $U \in \mathcal{U}$ - this is so called Lebesgue number for the covering $\mathcal{U}$ (in case you don't remember - its existance is proved in the course Topology I).
According to the lemma 1.2.14 there exists $n \in \mathbb{N}$ such that

$$\operatorname{mesh} K^{(n)} < \varepsilon/2.$$

Now let v be a vertex of $K^{(n)}$. Suppose $x, y \in \operatorname{St}(v)$. Then there exist simplices $\sigma, \sigma' \in K^{(n)}$ such that $x \in \operatorname{int} \sigma, y \in \operatorname{int} \sigma'$ and $v \in \sigma \cap \sigma'$. The application of the triangle inequality then shows that

$$|x - y| \leq |x - v| + |y - v| < 2 \operatorname{mesh} K^{(n)} < \varepsilon.$$

Hence $\operatorname{St}(v) \subset U$ for some $U \in \mathcal{U}$ and the proposition is proved. $\square$

Corollary 1.2.16. *Suppose X is a compact polyhedron and $\mathcal{U}$ an open covering of X. Then there exists a triangulation of X which is finer then $\mathcal{U}$.*

Proof. Obvious from the previous proposition. $\square$

Previous corollary is actually true for arbitary polyhedron, but the proof is more difficult. In general case the barycentric subdivision cannot be used
.

As an application we will prove approximation theorem for continuous mappings between polyhedra.
Suppose K, K' are simplicial complexes and $g\colon |K| \to |K'|$ is a mapping. Mapping g is called **simplicial** if for every $\sigma \in K$ there exists $\sigma' \in K'$ such that $g(\sigma) \subset \sigma'$ and $g|\sigma\colon \sigma \to \sigma'$ is simplicial. Clearly simplicial mapping is completely determined by the images $g(v)$ of vertices of K, which are also vertices of K'. If $\{v_0, \dots, v_n\}$ are vertices of a simplex in K, then $\{g(v_0), \dots, g(v_n)\}$ span a vertex of K'.
Conversely suppose g is a mapping defined on the set of vertices of K which

satisfies condition
1)If $\{v_0, \ldots, v_n\}$ are vertices of a simplex in K, then $\{g(v_0), \ldots, g(v_n)\}$ are vertices of a simplex in K'.
Then g can be extended to the unique simplicial mapping $g\colon |K| \to |K'|$ (exercise 1.15).

Every simplicial mapping is continuous with respect to weak topologies (exercise 1.9b).

Definition 1.2.17. *Suppose K, K' are simplicial complexes and $f\colon K \to K'$ is a continuous mapping. A simplicial mapping $g\colon |K| \to |K'|$ is called* **a simplicial approximation of** f *if*
$f(x) \in \operatorname{int}\sigma$ *implies* $g(x) \in \sigma$ *for every* $x \in |K|$.

One of the main reasons simplicial approximations are considered is the following. Recall that the mappings $f, g\colon X \to Y$ (where X and Y are topological spaces) are called **homotopic** (written as $f \simeq g$) if there exists continuous $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Such an F is called **a homotopy** from f to g.
If $A \subset X$ is such that $F(x,t) = f(x) = g(x)$ for all $t \in I$, the mapping F is called **a homotopy relative to** A and f and g are said to be **homotopic relative to** A (written as $f \simeq g$ rel A).

Lemma 1.2.18. *Suppose simplicial mapping $g\colon |K| \to |K'|$ is a simplicial approximation to $f\colon |K| \to |K'|$. Denote $A = \{x \in |K| \mid f(x) = g(x)\}$. Then f and g are homotopic relative to A.*

Proof. We will consider only the case of finite complexes.
By the definition of the approximating mapping $f(x)$ and $g(x)$ belong to the same simplex σ of K' for every $x \in |K|$. Hence the line segment between $f(x)$ and $g(x)$ lies interely within $|K'|$, so the mapping $F\colon |K| \to I$,

$$F(x,t) = tf(x) + (1-t)g(x)$$

is well-defined. It is clearly continuous, since operations of addition and scalar multiplication are continuous in finite-dimensional spaces, whose topology induces weak topology on finite complexes (lemma 1.2.5).

In general case the continuity of F is not so simple to show and requires the proof that the product topology of $|K| \times I$ is coherent with the family $\{\sigma \times I\}_{\sigma \in K}$. This is suprisingly untrivial. Since we won't need the general case anyway, we skip the proof of it. $\square$

The following is the useful characterization of a simplicial approximation. Notice that in this formulation the mapping g defined on a set of vertices of K is **not** assumed to be simplicial apriori, which can be convinient in practice.

Lemma 1.2.19. *Suppose $f\colon |K| \to |K'|$ is continuous and a mapping g defined on the set of vertices of K with values in the set of vertices of K' is given. Then g can be extended to a simplicial approximation of f (in a unique way) if and only if*

$$f(\mathrm{St}(v)) \subset \mathrm{St}(g(v))$$

for every vertex $v \in K$.

Proof. The proof that simplicial approximation satisfies the condition is left as an exercise (1.16).

Suppose g satisfies the condition. Let us first prove that g can be extended to a simplicial mapping. Suppose $\{v_0, \ldots, v_n\}$ is a set of vertices of a simplex $\sigma \in K$. Let b be a barycentre of σ. Then $b \in \mathrm{St}(v_i)$ for all $i = 0, \ldots, n$. It follows that $c = f(b) \in \cap_{i=0}^{n} \mathrm{St}(g(v_i))$. Let σ' be a unique simplex that contains c as an interior point. By the definition of star and the fact that interiors of different simplices do not intersect it follows that $g(v_i)$ is a vertex of σ' for every $i = 0, \ldots, n$. In particular $\{g(v_0), \ldots, g(v_n)\}$ are vertices of a simplex in K' (a face of σ').
Hence g can be extended to a simplicial mapping $g\colon |K| \to |K'|$ in a unique way. It remains to show that it is a simplicial approximation. Suppose $f(x) \in \operatorname{int} \sigma \in K'$. Let $\{v_0, \ldots, v_n\}$ be the vertices of a unique simplex of K, that contains x as an interior point. Then $x \in \mathrm{St}(v_i)$ for all $i = 0, \ldots,$ so $f(x) \in \mathrm{St}(g(v_i))$. As above we see that $g(v_i)$ is a vertex of σ for every $i = 0, \ldots, n$. Since g is simplicial, it follows that $g(x)$ is a convex combination of $g(v_i)$, hence belongs to σ as well. The claim is proved. $\square$

Now we can state and prove the final main result of this section.

Proposition 1.2.20. *Suppose K is a finite simplicial complex, K' is a simplicial complex and $f\colon |K| \to |K'|$ is continuous. Then there exists $n \in \mathbb{N}$ such that f has a simplicial approximation $g\colon |K^{(n)}| \to |L|$.*

Proof. Consider the open covering

$$\mathcal{U} = \{f^{-1}(\mathrm{St}(v') \mid v' \text{ is a vertex of } K'\}$$

of $|K|$. By the proposition 1.2.15 there exists $n \in \mathbb{N}$ such that $K^{(n)}$ is finer than the covering $\mathcal{U}$. This means that for every vertex v of $K^{(n)}$ there exists a vertex $g(v)$ of K' such that

$$f(\mathrm{St}(v)) \subset \mathrm{St}(g(v)).$$

By the previous lemma g can be extended to a simplicial approximation of f. ☐

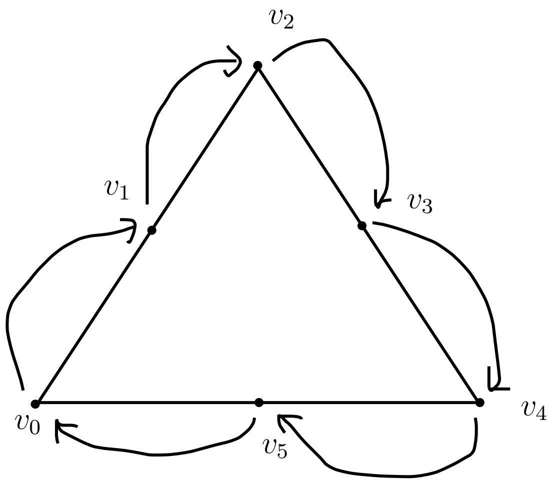
It is a well-known fact (see Topology II or proof yourself) that the homotopy relation $f \simeq g$ is an equivalence relation on the set of all continuous mappings $X \to Y$ (X, Y fixed topological spaces). Corresponding quotient set will be denoted $[X, Y]$.

Corollary 1.2.21. *Suppose X and Y are compact polyhedra. Then the set $[X, Y]$ is countable.*

Proof. Choose finite simplicial complexes K, K' such that $X = |K|, Y = |K'|$ (up to a homeomorphism). Let $f\colon X \to Y$ be an arbitrary continuous mapping. By the proposition **??** there exists $n \in \mathbb{N}$ such that f has a simplicial approximation $g\colon |K^n| \to |K'|$. By the lemma 1.2.18 g is homotopic to f.
For every fixed $n \in \mathbb{N}$ there exists only a finite amount of simplicial mappings $g\colon |K^{(n)}| \to |K'|$, since such a mapping is completely determined by the way it maps vertices to vertices, and there is only a finite amount of vertices in both complexes.
Since the countable union of finite sets is countable, the claim follows. ☐

Examples 1.2.22. *1. Later we will prove that $[S^n, S^n]$ is infinitely countable for $n > 0$. Fix a triangulation of $S^n = |K|$. For every $m \in \mathbb{N}$ $K^{(m)}$ and K are finite complexes, so there exists only a finite amount of possible simplicial mappings $g\colon |K^{(m)}| \to |K|$.*
Since $[|K|, |K|]$ is infinite, for every fixed $m \in \mathbb{N}$ there must be a continous $f\colon |K^{(m)}| \to |K|$ which does not have a simplicial approximation $g\colon |K^{(m)}| \to |K|$. Hence it is necessary to consider arbitrary $m \in \mathbb{N}$ in the proposition1.2.20.

2. Consider the boundary of the equilateral triangle σ as a 2-simplex with vertices v_0, v_2, v_4. For odd $i = 1, \ldots 5$ denote by v_i the barycentre of the 1-simplex $[v_{i-1}, v_{i+1}]$, where we identify $v_6 = v_0$.
Let $K = K(\partial\sigma)$. Let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K'| \to |K'|$ defined by $f(v_i) = v_{i+1}$. f can be thought of as a 60° "rotation" (under the canonical projection homeomorphism to the sphere).



Now as a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation. As a mapping $f\colon |K'| \to |K|$ f has exactly 8 simplicial appoximations - under any approximation g barycentres v_i (odd i) must be mapped to v_{i+1} and for even v_i there are exactly two choices for $g(v_i)$ - v_i or v_{i+1}. The verification of these claims is left as an exercise (1.17).

Using simplicial approximation-theorem one can easily prove the following interesting topological result.

Theorem 1.2.23. *Suppose $m < n$ and $f\colon S^m \to S^n$ is a continuous mapping. Then f is* **homotopically trivial** *i.e. homotopic to a constant mapping.*

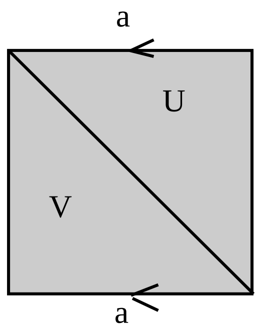
Proof. Exercise 1.18. □

1.3 Δ-complexes

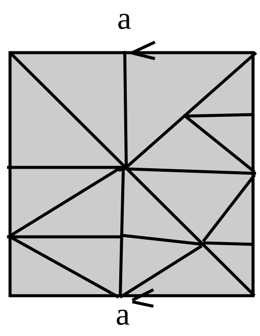
Simplicial complexes provide a classical way to study polyhedrons, which is useful both theoretically as well as in practic. However in some circumstances the simplicial approach is "too regular" and rigid. Many spaces that occur in practice can be triangulated, but the triangulation might be too complicated for practical purposes. For example a projective space $\mathbb{R}P^2$ is a polyhedron, but to represent it one needs a simplicial complex that has at least 10 triangles, 15 edges, and 6 vertices.

That is why we briefly introduce the notion of Δ-complex (pronounced: "Delta-complex"), which is more flexible, "modern" way to use simplicial approach. It does have some drawbacks as well, but suits very well for the first introduction and illustation of the homological methods.

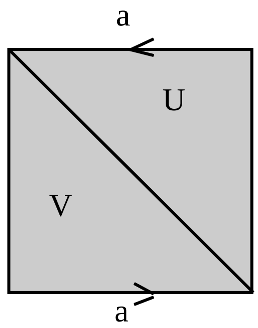
Before going into complicated formalities that us first grasp the idea via simple examples. Let us start with the same square devided into 2 triangules along the diagonal.



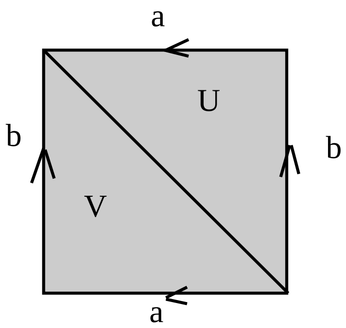
This is an excellent way to triangulate a square, but what if we glue together, say, horizontal sides of the square (both indicated by the letter 'a' in the picture), thus obtaining a hollow tube. It is very tempting to represent the space thus obtained as a sort of a simplicial complex, where also sides 'a' are common sides of the triangle. Now this won't be a simplicial complex in a strict sence we definied it to be in the previous section, since now we have two triangles, whose intersection is not a common side, but a union of two common sides. Nevertheless it provides a very simple combinatorical description of our space. We could introduce a " subdivision" that would be an honest simplicial complex, whose polyhedra is our tube (see the picture below), but it would be more complicated and have more simplices. Also, the simple geometrical intuition and naturality is lost.



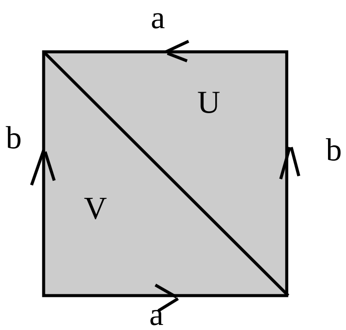
Let us continue with the same ideas around our square. If we again glue together horizontal sides but changing direction of the one of them, we obtain a familiar Mobius band. Again we can think of it as a union of two triangules with two common sides - this time the way the sides are identified is just slightly different.



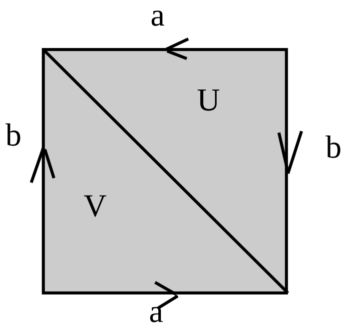
If we identify horizontal sides together and vertical sides together, both having the same orientation, we obtain a torus. This time the intersection of two triangles consists of their mutual boundary with some interesting identifications - in fact it is easy to see that all four vertices are now identified together and "1-simplices" of this " complex" form 3 circles glued in a point (which corresponds to the glued vertices).



If we "twist" one of the sides, for example leave the identification of two vertical sides as above, but glue horizontal sides with opposite orientations, we obtain a Klein's bottle. Once again all vertices are identified at one point and "1-skeleton" consists of 3 circles glued together at this point.



If both sides are identified "with a twist" the resulted space is a projective space $\mathbb{R}P^2$.



So we see that by loosing up the strict rules of simplicial complexes a little, we immediately obtain simple combinatorial description of some interesting well-known spaces. These examples give us enough courage to formalise these ideas.

For the technical reasons we choose to formalise new notion in terms of ordered simplices. We denote an ordered simplex with vertices $v_0 < v_1 < \ldots < v_n$ by an $(n+1)$-tuple $(v_0, \ldots, v_n)$. Recall that if $(w_0, w_2, \ldots, w_n)$ is another ordered simplex of the same dimension there is a unique simplicial mapping $f\colon (v_0, \ldots, v_n) \to (w_0, \ldots, w_n)$ that preserves ordering, namely the one determined by $f(v_i) = w_i$ for all $i = 0, \ldots, n$. This mapping is then necesarily a homeomorphism.

We also adopt the following notation: if $(v_0, \ldots, v_n)$ is an ordered simplex as above, its ith face $(v_0, \ldots, \widehat{v_i}, \ldots, v_n)$ is denoted by $\partial^i(\sigma)$.

Definition 1.3.1. A Δ-complex K *consists of the following data.*
1) A collection $\{\sigma_j\}_{j\in I}$ of ordered simplices, such that every face (with induced natural order) of a simplex in K is also a simplex in K. It is not required that all simplices lie in the same vector space.

2) An equivalence relation $\sim$ defined on the set K_n of ordered n-simplices of K for every $n \in \mathbb{N}$. We assume that these relations respect faces in a natural way - if $\sigma \sim \sigma'$ also $\partial^i(\sigma) \sim \partial^i(\sigma')$ for all $i = 0, \ldots, \dim \sigma$.

Of course what we are really interested in is the space obtained from this data.
We define a **polyhedron** $|K|$ of the given Δ-complex K as follows. First we form the disjoint topological union of all simplices in K,

$$Z = \bigsqcup_{j\in I} \sigma_j.$$

Next we do the identifications of 2-types,
1) If $\sigma' < \sigma$ we identify σ' with its copy in σ (one of the faces), in an obvious

way.
2) If $\sigma \sim \sigma'$ let $f\colon \sigma \to \sigma'$ be a unique order-preserving simplicial bijection. Then identify $x \in \sigma$ and $f(x) \in \sigma'$.

These identifications define an equivalence relation $\sim$ on Z.
Finally define

$$|K| = Z/\sim$$

to be a quotient space, equipped with a quotient topology. This space is called a polyhedron of the Δ-complex K. More generally we could say that a space X is a polyhedron if it is homeomorphic to a polyhedron of some Δ-complex. This is not in contradiction with our previous terminology - it can be proved that every Δ-complex is trianguable, although we won't go into proving that. Hence we don't obtain new spaces, but we do obtain a more economical and efficient way to desribe our spaces with combinatorial data.

The image of a simplex $\sigma = (v_0, \dots, v_n) \in J$ in the quotient space $|K|$ is denoted by $[v_0, \dots, v_n]$ and is called a **geometric n-simplex** of $|K|$. Geometrically it looks like a simplex, but with some faces possibly identified. For example S^1 can be represented as a 1-simplex $[v, v]$ with ith endpoints identified.
The set of geometric simplices is essentialy the same as the quotient set $K_n/\sim$, so we will denote it as $K_n/\sim$.

We also introduce the notion of a characteristic mapping of σ.
Let Δ_n be the standard n-simplex, considered as an ordered simplex $(e_0, \dots, e_n)$. Let $\alpha\colon \Delta_n \to \sigma$ be the unique simplicial homeomorphism that preserves the order of vertices. Let $i\colon \sigma \to Z$ be a natural imbedding of σ in the disjoint union of all simplices. Finally let $\pi\colon Z \to |K|$ be a quotient map. We define a characteristic mapping of σ denoted by

$$f_\sigma\colon \Delta_n \to |K|$$

to be the composition $\pi \circ i \circ \alpha$. If $\sigma \sim \sigma'$, then obviously

$$f_\sigma = f'_\sigma.$$

Otherwise f_σ and f'_σ are not the same mapping. Hence there is a bijective correspondence between the set of all geometric simplices of $|K|$ and the set of all characteristic mappings.
From the definition of $|K|$ it follows that all identifications inside a given simplex σ happen on the boundary. Hence

Lemma 1.3.2. *The restriction $f_\sigma|\operatorname{int}\Delta_n$ of the characteristic mapping f_σ to the interior of Δ_n is injective, in fact a homeomorphism to its image $f_\sigma(\operatorname{int}\Delta_n)$, which we call* **the interior** *of the geometric simplex $f_\sigma(\Delta_n)$. $|K|$ is a disjoint union of these interiors.*
The topology of $|K|$ is co-induced by the set of characteristic mappings $\{f_\sigma\}_{\sigma\in K}$.

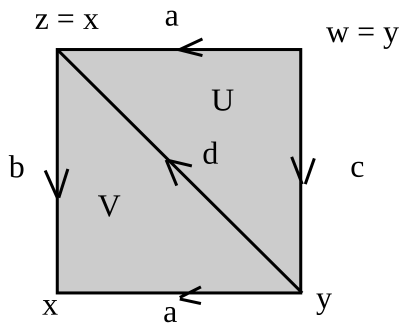
Proof. Exercise. □

A Δ-subcomplex L of K is defined in an obvious manner - the collection of its simplices must be closed under all faces and two simplices in L are identified in L if and only if they are identified in K.
In this case $|L|$ imbedds as a subspace of $|K|$ in an obvious way and is closed in it (exercise).

Example 1.3.3. *Suppose K is a Δ-complex and $n\in\mathbb{N}$. Denote by K^n a subcomplex generated by all simplices of K with dimension $\leq n$. Obviously this set is closed under faces and identifications, so it really is a subcomplex. It is called* **the n-skeleton** *of K. Correspoding subspace $|K^n|$ of $|K|$ is called* **the n-skeleton** *of $|K|$.*
It is clear that K^0 is just a collection of 0-simplices, which are isolated points, so $|K^0|$ is also a disjoint union of isolated points (some might be identified but this does not effect the conlusion). In other words K^0 is a discrete space. The elements of $|K^0|$ are called **the vertices** *of the polyhedron $|K|$.*
The 1-simplices of $|K|$ are called **the edges***.*

When one tries to represent a given space as a polyhedron of a Δ-complex, it is important to pay attention to the ordering - remember that all simplices must be ordered and whenever you want to identify two faces of different simplices identification **must preserve** ordering.

Example 1.3.4. *Let us illustrate this with the example of hollow tube from the beginning of this section.*
All we have to do is to choose the ordering of the vertices of the both triangles, so that it is compatible with the identifications of faces. The picture below shows one possibility. The ordering of vertices is indicated by an arrow on every 1-simplex, which goes from the smaller vertex to the greater vertex.



Suppose the corners of this square are the points $x = (0,0), y = (1,0), z = (0,1), w = (1,1)$ *of the plane. Then the* Δ*-complex consists of two* 2*-simplices* U *and* V *with ordering of vertices* $V = (y,z,x)$ *and* $U = (w,y,z)$*, and their faces -* (y,x) *and* (w,z)*, which are identified (and called 'a'),* $b = (z,x)$*,* $c = (w,y)$ *and* $d = (y,w)$*. Due to identification there are only two vertices -* x *and* y*, since after the identifications take place we obtain that* $z = x$ *and* $w = y$ *.*

As a further example/exercise reader should go through all examples in the beginning of this section (except for the tube which is already checked) and give a formal description of corresponding Δ-complexes. Remember to pay attention to the ordering!

Every polyhedron of a simplicial complex K can be considered as a polyhedron of a Δ-complex in a natural way. We do need to order every simplex in a consistent way though, but this can be always done - just choose some linear ordering on the set of all vertices of K. You might need the Axiom of Choice for large cases, but you do believe in the Axiom of Choice, don't you? :)

It follows that all the constructions made for Δ-complexes work for simplicial complexes as well, in particular a simplicial homology defined in the next section.

1.4 Exercises

1.4.1 Simplices

1. Consider the pairs $(V, \{v_1, \ldots, v_n\})$, where V is finite-dimensional vector space and $\{v_1, \ldots, v_n\}$ is a fixed basis of V. Thus for every $n \in \mathbb{N}$ the pair $(\mathbb{R}^n, \{e_1, \ldots, e_n\})$ is an example of such pair. Moreover for every pair $(V, \{v_1, \ldots, v_n\})$ there is a unique linear bijection $f\colon V \to \mathbb{R}^n$

such that $f(v_i) = e_i$ for all $i \in \{1, \dots, n\}$.
a) Assign to a pair $(V, \{v_1, \dots, v_n\})$ unique topology such that f as above is a homeomorphism. Prove that $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ are continuous with respect to this topology.
Suppose $(W, \{w_1, \dots, w_m\})$ is another pair and $l\colon V \to W$ is linear. Deduce that l is continuous.
b) Deduce that the topology so assigned to V does not depend on the chosen basis $\{v_1, \dots, v_n\})$ (apply a) to the identity mapping).

2. Suppose $A \subset V$ is a non-empty subset. Prove that A is affine if and only if there is $v \in V$ and a linear subspace $W \subset V$ such that $A = x + W$. Moreover show that in this case W is unique.

3. a) Show that an affine/convex set A is closed under affine/closed combinations. In other words prove that if $a_1, \dots, a_n \in A$, $r_1, \dots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and in convex case also $r_i \geq 0$ for all $i = 1, \dots, n$, then
$$r_1 a_1 + \ldots + r_n a_n = x \in A.$$
b) Suppose $A \subset V$. Prove that
$$\operatorname{aff}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_1 + \ldots + r_n = 1\},$$
$$\operatorname{conv}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_i \geq 0, r_1 + \ldots + r_n = 1\}.$$
c) Suppose $f\colon C \to C'$ is an affine mapping between convex sets. Prove that
$$f(r_1 a_1 + \ldots + r_n a_n) = r_1 f(a_1) + \ldots + r_n f(a_n),$$
if $a_1, \dots, a_n \in A$, $r_1, \dots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and $r_i \geq 0$ for all $i = 1, \dots, n$.

4. Prove that the set of vertices of a simplex is uniquely determined by the simplex. (Hint: show that a point is not a vertex if and only if it is a midpoint of an interval contained entirely in the simplex).

5. Let V be a finite-dimensional vector space.
a) Suppose $A \subset V$ and $\{v_0, \dots, v_n\}$ is a maximal (with respect to inclusion) affinely independent subset of A. Prove that $\operatorname{aff}(A) = \operatorname{aff}(\{v_0, \dots, v_n\})$.
b) Suppose $C \subset V$ is convex and non-empty. Prove that C has a non-empty interior with respect to $\operatorname{aff}(C)$. (Hint: use a) and notice that the simplex spanned by $\{v_0, \dots, v_n\}$ is a subset of C.)

6. Show that the standard n-simplices defined by

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\},$$

$$\Delta'_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} x_i = 1\}$$

are compact Hausdorff spaces (as subsets of Euclidean spaces).

7. Suppose $C \subset \mathbb{R}^n$ is a closed bounded convex set and 0 is the interior point of C. Let $f \colon \partial C \to S^{n-1}$, $f(x) = x/|x|$ and assume known that f is a homeomorphism.
Prove that $G \colon \overline{B}^n \to C$ defined by

$$G(t) = \begin{cases} |t| \cdot \left(f^{-1} \frac{t}{|t|}\right) \text{ if } t \neq 0, \\ 0, \text{ if } t = 0 \end{cases}$$

is a homeomorphism.

1.4.2 Simplicial complexes.

8. Suppose V is a vector space. Show that the collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is a simplicial complex if and only if
1) For every simplex σ in K its every face also belongs to K.
2') For every $x \in \bigcup_{i \in I} \sigma_i$ there is a unique $i \in I$ such that x is an interior point of the simplex σ_i.

9. Suppose K is a simplicial complex and X is a topological space.
Prove that a mapping $f \colon |K| \to X$ is continuous with respect to the weak topology of $|K|$ if and only if the restriction of f to every simplex $\sigma \in K$ is continuous.
b) Conclude that every simplicial mapping $f \colon |K| \to |K'|$ (K and K' simplicial complexes) is continuous.

10. Suppose L is a subcomplex of a simplicial complex K. Show that
a) The weak topology on the simplicial complex $|L|$ is the same as the relative topology on $|L|$ induced by the weak topology of $|K|$.
b) $|L|$ is closed in $|K|$.

11. Suppose $x \in |K|$.
a)Define $L = \{\sigma \in K | x \notin \sigma\}$. Show that L is a simplicial complex and

$$|K| \setminus |L| = \mathrm{St}(x).$$

Conclude that $\mathrm{St}(x)$ is an open neighbourhood of x in $|K|$.
b)Suppose $x \in |K|$ and all the vertices of $\mathrm{car}(x)$ are $v_0, \dots, v_n$.
Prove that

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid \mathrm{car}(x) < \sigma\} = \bigcup\{\mathrm{int}\,\sigma \mid v_0, \dots, v_n \text{ are vertices of } \sigma\}.$$

and

$$\mathrm{St}(x) = \bigcap_{i=0}^{n} \mathrm{St}(v_i).$$

12. Suppose K' is a subdivision of K. Prove that the Euclidean topologies defined by K' and K on $|K'| = |K|$ coincide

13. a) Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{v_0, \dots, v_n\}$. Prove that

$$\mathrm{diam}\,\sigma = \max\{|v_i - v_j|\},$$

where $|\cdot|$ is a standard norm on $\mathbb{R}^m$.
b) Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in a first barycentric division $K^{(1)}$, with vertices $\{b(\sigma_0), b(\sigma_1), \dots, b(\sigma_n)\}$, where $\sigma_0 < \dots < \sigma_n = \sigma \in K$. Prove that

$$\mathrm{diam}\,\sigma' \leq \frac{n}{n+1}\,\mathrm{diam}\,\sigma$$

14. Let $L \subset K$ be a subcomplex of a simplicial complex K. We say that L is **full** in K if it satisfies the following condition. Suppose $a_0, \dots, a_n$ are vertices in L that span a simplex σ in K. Then $\sigma \in L$.
a) Give an example of a simplicial pair (K, L) such that L is not full in K.
b) Suppose (K, L) is a simplicial pair. Prove that $L^{(1)}$ is full in $K^{(1)}$.

15. Suppose K, K' are simplicial complexes and a mapping g defined on the set of vertices of K is given, satisfying the following condition:

If $\{v_0, \ldots, v_n\}$ are vertices of a simplex in K, then $\{g(v_0), \ldots, g(v_n)\}$ are vertices of a simplex in K'.
Prove that g can be extended to a simplicial mapping $g\colon |K| \to |K'|$ in a unique way.

16. Suppose g is a simplicial approximation of the continuous mapping $f\colon |K| \to |K'|$. Show that

$$f(\mathrm{St}(v)) \subset \mathrm{St}(g(v))$$

for every vertex $v \in K$.

17. Consider the boundary of the equilateral triangle σ as a 2-simplex with vertices v_0, v_2, v_4. For odd $i = 1, \ldots 5$ denote by v_i the barycentre of the 1-simplex $[v_{i-1}, v_{i+1}]$, where we identify $v_6 = v_0$.
Let $K = K(\partial\sigma)$. Let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K^{(1)}| \to |K^{(1)}|$ defined by $f(v_i) = v_{i+1}$. Prove that as a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation, but as a mapping $f\colon |K^{(1)}| \to |K|$ f has exactly 8 simplicial appoximations. List all approximations.

18. a) Suppose $m \in \mathbb{N}$. Let K be an m-dimensional simplicial complex and K' be a simplicial complex whose dimension is $> m$. Show that every continuous mapping $f\colon |K| \to |K'|$ is homotopic to a mapping, which is not surjective (Hint: simplicial approximation).
b) Suppose $m < n$. Prove that any continuous mapping $f\colon S^m \to S^n$ is homotopic to a constant mapping.

19. Suppose $v_0, \ldots, v_n$ are vertices of the simplicial complex K. Show that they span a simplex of K if and only if

$$\cap_{i=1}^{n} \mathrm{St}(v_i) \neq \emptyset.$$

20. Suppose $f\colon |K| \to |K'|$ is a continuous mapping between polyhedrons. Let g and g' be simplicial approximations to f. Prove that for every simplex σ

$$g(\sigma) \cup g'(\sigma)$$

is a simplex in K'.

21. Suppose $f\colon |K| \to |K'|$ is a continuous mapping between polyhedrons, g is a simplicial approximation of f and $L \subset K$ is a subcomplex such that $f||L|$ is simplicial. Prove that $f||L| = g||L|$.

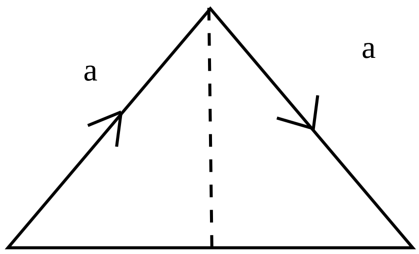
1.4.3 Δ-complexes

22. Suppose K is a Δ-complex and σ is an n-simplex of K.
Show that the restriction of the characteristic mapping $f_\sigma| \operatorname{int} \Delta_n$ to the interior of Δ_n is a homeomorphism to its image and $|K|$ is a disjoint union of the sets $\{f_\sigma(\operatorname{int} \Delta_n)\}$ (meaning that two sets are either the same or disjoint).
Prove that the topology of $|K|$ is co-induced by the set of charachteristic mappings $\{f_\sigma\}_{\sigma \in K}$.

23. Prove that the examples constucted from the square in the beginning of the section on Δ-complexes are indeed Δ-complexes (except for the first example of the tube, which is already checked in the lecture notes). Give an ordering on every simplex, pay attention to the comparatibility of the ordering and identifications!

24. Suppose L is a subcomplex of a Δ-complex K. Show that $|L|$ is a closed subspace of $|K|$ in an obvious way.

25. Suppose in an ordered triangle $[v_0, v_1, v_2]$ i.e. 2-simplex you identify two faces $[v_0, v_1]$ and $[v_1, v_2]$ (preserving the ordering, as usual). What familiar space is this quotient space homeomorphic with? (Hint: cut the triangle in half, as the picture indicates, making it a simplicial complex made up by two triangles, then do the identification, then glue triangles back. Drawing pictures might help! Remember to keep the track of the ordering.)



What if we identify sides $[v_0, v_1]$ and $[v_0, v_2]$ instead?