FUNKTIONAALIANALYYSI II, 2011
EXERCISES, SET 4
TO BE RETURNED ON TUESDAY DEC. 13th AT LATEST, PERSONALLY OR TO THE MAILBOX OF J.T.

1. Is the operator

$$L := (1 - |x|^2) \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$$

elliptic on the domain $\Omega = \{|x| < 1\} \subset \mathbb{R}^3$?

2. Is the operator

$$a)\ L := \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \quad , \qquad b)\ T := \Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \Big)^2$$

elliptic on $\Omega := \mathbb{R}^2$?

3. Let Ω be a bounded domain in $\mathbb{R}^n$ and assume $1 \leq n < p$, $\lambda := 1 - n/p$. Using Proposition 6.13, i.e. the existence of a continuous embedding

$$W_0^{1,p}(\Omega) \to C^{0,\lambda}(\bar{\Omega})$$

and proofs and methods of earlier results, prove Corollary 6.14: assume that $m \in \mathbb{N}$, $j \in \mathbb{N}_0$, $0 < \lambda \leq 1$, $1 \leq p < \infty$ and $(m - j - \lambda)p \geq n$. Then there exists a continuous embedding

$$W_0^{m,p}(\Omega) \to C^{j,\lambda}(\bar{\Omega}).$$

4. Let $\Omega :=]-1, 1[\subset \mathbb{R}$ and $1 < p < \infty$. Construct a continuous extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R})$. ("Extension operator" means that E must have the property $(Ef)(x) = f(x)$ for all $f \in W^{1,p}(\Omega)$, for almost every $x \in \Omega$, i.e. E extends the functions $f \in W^{1,p}(\Omega)$ to the whole set $\mathbb{R}$.) Hint. The case II.C') in Theorem 6.4 holds, so that we may assume $f \in W^{1,p}(\Omega)$ belongs to the space $C([-1, 1])$. In particular, $f(-1)$ and $f(1)$ are well defined. Define the extension of f so that $f(x) = 0$ for $|x| \geq 2$; but what do you do for $1 < |x| < 2$?
Make a guess (without proofs) about possible extensions $E^{(j)} : W^{j,p}(\Omega) \to W^{j,p}(\mathbb{R})$ for larger $j \in \mathbb{N}$.

5.–6. Let $1 \leq p < \infty$, $m \in \mathbb{N}$ and $p < q < np/(n - mp)$, where n is the dimension of the domain, $\mathbb{R}^n$. By constructing an explicit sequence of functions, show that the embedding $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, existing by Theorem 6.4. of the lectures, cannot be compact, i.e., the identity operator is not a compact operator, i.e., the unit ball of the Banach space $W^{m,p}(\mathbb{R}^n)$ is not a precompact subset of $L^q(\mathbb{R}^n)$, i.e. there exists a sequence $(f_l)_{l \in \mathbb{N}} \subset W^{m,p}(\mathbb{R}^n)$ such that $\|f_l\|_{p,m} = 1$ for all l but no subsequence converges in $L^q(\mathbb{R}^n)$.

7.–9. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a regular enough (at least C^2) boundary. Consider the following Neumann problem:

$$-\Delta u + u = f \quad \text{in the domain } \Omega$$
$$\partial_\nu u = 0 \quad \text{in } \partial\Omega,$$

where $f \in C(\bar{\Omega})$ is given and ∂_ν denotes the partial derivative in the direction of the outer normal vector of the boundary. By a *classical solution* we mean a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying the above equalities.

a) Let us define the *weak solution* as a function $u \in W^{1,2}(\Omega)$, which satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi$$

for all $\varphi \in W^{1,2}(\Omega)$; here "$\cdot$" denotes the inner product of $\mathbb{R}^3$. By using a relevant *Green formula* show that a classical solution to the above Neumann problem is always a weak solution.

b) Prove the existence and uniqueness of the weak solution using the Lax–Milgram theorem.