FUNKTIONAALIANALYYSI II, 2011
EXERCISES, SET 3
TO BE RETURNED ON WED. NOVEMBER 23rd AT LATEST, PERSONALLY OR TO THE MAILBOX OF J.T.

1. Let $\Omega :=]0,\infty[\subset \mathbb{R}$. Show that the following inclusions are *strict*, i.e., the spaces are not the same: $C^1(\bar\Omega) \subset C^1_B(\Omega) \subset C^1(\Omega)$.

2. Show that the space $C(\bar{\mathbb{R}})$ is not dense in $L^\infty(\mathbb{R})$.

3. For all λ, $0 < \lambda < 1$, construct a function $f : \mathbb{R} \to \mathbb{R}$, which is an element of the Hölder–space $C^{0,\lambda}(\bar{\mathbb{R}})$ but not that of $C^1(\bar{\mathbb{R}})$.

4. Let $a > 0$ and $b > 0$, and $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and

$$f(x) := x^{-a} \quad , \quad g(x) := x^{-b}$$

for $|x| \le 1$, and $f(x) = g(x) = 0$ for $|x| > 1$.
What does the Young inequality imply about the existence of the function $f * g$ as an element in the space $L^r(\mathbb{R})$ for different a, b and $1 \le r \le \infty$?

5. Show that the operator P defined just before Theorem 5.12 in the lecture notes is an isometry from the Sobolev space onto a closed subspace of X.

6. Show by an example, that $W^{1,1}_0(\Omega) \ne W^{1,1}(\Omega)$ for $\Omega := \{|x| < 1\} \subset \mathbb{R}^2$.

7. Using the definition, show that the open unit ball of $\mathbb{R}^2$ has the cone property.

8. For which values of $a \in \mathbb{R}$ the function $|x|^a = (x_1^2 + x_2^2)^{a/2}$ belongs to the Sobolev space $W^{1,p}(\Omega)$, where Ω is the open unit ball $\{|x| < 1\}$ of $\mathbb{R}^2$ and $1 \le p \le \infty$?

9. The same as in the previous problem, for the function $|x|^a = (x_1^2 + x_2^2 + x_3^2)^{a/2}$, the Sobolev space $W^{1,p}(\Omega)$, and Ω as the open unit ball of $\mathbb{R}^3$.