FUNKTIONAALIANALYYSI II, 2011
EXERCISES, SET 2

1. Let us consider here *simple functions of intervals* χ in $\mathbb{R}$, which are constants on some subintervals of $\mathbb{R}$, e.g. $\chi(x) = 2$ for $1 \leq x \leq 4$ and $\chi(x) = -1$ for $4 < x \leq 5$ and $\chi(x) = 0$ elsewhere.
Find a sequence of simple functions which converges (weakly) in $\mathcal{D}'(\mathbb{R})$ to the distribution a) δ_0, b) δ_0'.
Is it true that in $\mathcal{D}'(\mathbb{R})$,

$$\lim_{n\to\infty} n\delta_{-1/n} - n\delta_{1/n} = 2\delta_0'.$$

2. Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be sequences of, say, functions in $L^1(\mathbb{R})$ such that $f_n \to \delta_a$, $g_n \to \delta_b$ in $\mathcal{D}'(\mathbb{R})$, where a, $b \in \mathbb{R}$ are fixed. Let us assume in addition that the $L^1(\mathbb{R})$-norms of all functions are bounded by some constant $C > 0$. Prove that

$$\lim_{n\to\infty} f_n \otimes g_n \to \delta_{(a,b)}$$

in $\mathcal{D}'(\mathbb{R}^2)$.

3. Give examples of sequences of simple functions of *rectangles* and *discs* of $\mathbb{R}^2$, which converge to $\delta_{\bar{0}}$ in $\mathcal{D}'(\mathbb{R}^2)$. Problem 2 may be of some use.

4. Write the distribution $T \in \mathcal{D}'(\mathbb{R})$,

$$T := \delta_5 - \frac{d\delta_0}{dx}$$

as a derivative of a continuous function on $\mathbb{R}$.

5. Let $Y : \mathbb{R} \to \mathbb{C}$ be the step function. Show that $d\delta_0/dx * Y = \delta_0$ and that $1 * d\delta_0/dx = 0$. Calculate

$$1 * \left(\frac{d\delta_0}{dx} * Y\right) \qquad \text{and} \qquad \left(1 * \frac{d\delta_0}{dx}\right) * Y. \tag{0.1}$$

This seems to violate the associative law. What's wrong?

6. Let $f \in C^\infty(\mathbb{R}^n)$, and let $\gamma_{k,m}$ be as in (4.1) of the lecture notes. Show that the condition "$\gamma_{k,m}(f) < \infty$ for all k and m" is equivalent to the condition

$$\text{“} \lim_{|x|\to\infty} |x|^k |D^\alpha f(x)| = 0 \tag{0.2}$$

for all k and α".

7. Using the definition of the seminorms $\gamma_{k,m}$, show that the linear operator T,

$$(T\varphi)(x) = 5\sin(x)\varphi(x) + \varphi(2x) \tag{0.3}$$

is continuous $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

8. Show that the function e^{ax} is not a tempered distribution on $\mathbb{R}$, if $a \neq 0$ is a constant.

9. Show that the operator $G : T \mapsto \sin xT$ is sequentially continuous $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, when this space is endowed with the weak topology. Sequential continuity means

that if the sequence $(T_j)_{j=1}^{\infty}$ satisfies $T_j \to T$ weakly in $\mathcal{S}'(\mathbb{R})$ as $j \to \infty$, then $GT_j \to GT$ weakly in $\mathcal{S}'(\mathbb{R})$.

10. Calculate the Fourier–transforms of the tempered distributions x^k, $k \in \mathbb{N}$ on $\mathbb{R}$. Also calculate the Fourier–transform of the polynomial P of two variables, $P(x) := x_1^2 x_2$, where $x = (x_1, x_2) \in \mathbb{R}^2$.

11. Show that if $f : \mathbb{R}^n \to \mathbb{C}$ and if there exist constants $C > 0$ and $a > n + 1$ such that

$$|f(x)| \leq \frac{C}{(1 + |x|)^a} \tag{0.4}$$

then $\mathcal{F}f$ is at least m times differentiable for $m < a - n - 1$. (Differentiate under the integral sign.) This is an indication of the important basic intuition that the more rapidly f vanishes at the infinity, the more smooth is its Fourier tranformation. Conversely, if $\hat{f}$ vanishes at a certain rate at infinity, the f must have a corresponding amount of smoothness.