FUNKTIONAALIANALYYSI II, 2011
EXERCISES, SET 1

1. Prove that the space $\mathcal{D}(\mathbb{R})$ is not sequentially complete when endowed with the topology of $C^\infty(\mathbb{R})$.

2. Let us consider the function $f \in C^\infty(\mathbb{R}^3)$ of three real variables; denote $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Calculate the partial derivative $D^\alpha f$ for $\alpha = (1,1,0)$ and $(0,1,2)$, and for $f =$ a) $x_1 e^{x_2 + x_3}$, b) $1/(1 + x_2^2 + x_3^2)$.

3. Show that the linear mapping T is continuous from the space $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$, if a) $Tf(x) := f(x+3)$, b) $Tf(x) := \sin x f(x)$. (Here $f \in C^\infty(\mathbb{R})$ and $x \in \mathbb{R}$).

4. Example 2.14, problems (2.33) (which of course reads as $\varepsilon_n \to 0$) and (2.34). You may also consider the other examples (2.35), (2.36), though you may need to improve the proof of Th. 2.13.

5. Show that $\delta_0 \in \mathcal{D}'(\mathbb{R})$ is not a continuous positive function, i.e., $\delta_0 \neq I(f)$ for any continuous positive function f, for the embedding $I : L^1_{\text{loc}}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ constructed in the lecture notes. (Example b) on p.6). Can you actually show that δ_0 is not a locally integrable function?

6. Show that in $\mathcal{D}'(\mathbb{R}^2)$ we have

$$\Delta \log \frac{1}{r} = -2\pi \delta_0.$$

Here $r = |x|$, $x \in \mathbb{R}^2$.

7. Prove that in $\mathcal{D}'(\mathbb{R}^n)$, $n > 3$,

$$\Delta \frac{1}{r^{n-2}} = -(n-2)\lambda_n \delta_0,$$

where λ_n is the area of the unit ball of $\mathbb{R}^n$.

8. Prove that the sum

$$T = \sum_{j=1}^{\infty} \frac{\partial^j \delta_j}{\partial x^j}$$

converges in $\mathcal{D}'(\mathbb{R})$. Here δ_j is the Dirac measure of the point j. What is the order of the distribution T? Is it compactly supported?

9. Prove Theorem 2.20 of the lecture notes.

10. Show that if $f \in C^\infty(\mathbb{R})$ and $f(0) = 0$, then $f\delta_0 = 0$ in the space $\mathcal{D}'(\mathbb{R})$. In particular, $x\delta_0 = 0$.

11. Let $T \in \mathcal{D}'(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$. Is the following identity true (where f' is the classical derivative):

$$\frac{d(fT)}{dx} = f'T + f\frac{dT}{dx} \quad ?$$