EVOLUTION AND THE THEORY OF GAMES

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Transcript of my notes of 17-11-2011 and 22-11-2011

▶ *The beginning of the lecture of 17 November was a continuation on the War of Attrition. This part of the lecture was put together with the lecture notes of 15 November and is not repeated here.*

23. In the original Hawk-Dove game with payoff matrix

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	$\frac{1}{2}R$, $\frac{1}{2}R$

the strategy $x = (1, 0)$ (i.e., play H with probability one and D with probability zero) is an ESS if $R > C$, and $x = (R/C, 1 - R/C)$ is an ESS if $R \leq C$. The predicted frequency of escalated fights in the population (i.e., the frequency of H×H-contests) is therefore one if $R > C$, and R^2/C^2 if $R \leq C$.

The modified Hawk-Dove game in which we re-calculated the payoffs for the display contest (i.e., between Doves) has the payoff matrix

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	0 , 0

where the strategy $x = (1, 0)$ is an ESS if $R > C$, and $x = (2R/(C + R), (C - R)/(C + R))$ is an ESS if $R \leq C$. In the latter case, the predicted frequency of escalated fights is even higher than in the original Hawk-Dove game. Yet, escalated fights, although they do occur, are not at all *that* common in reality as suggested by the results above. To explain why, we introduce the notion of an asymmetric game.

24. Until now, the evolutionary games we considered were symmetric in the sense that the players were interchangeable: they had the same strategy set and the same payoffs. This is often and maybe even typically not the case: one player may be stronger than the other; the value of the resource or the expected costs may be different for the two players; in some games one player is male and the other female, and so forth. Such asymmetries may affect the payoffs as well as the strategy sets in a way that is different for the two players. What we need is an ESS solution concept that applies to such asymmetries.

The idea is to use so-called 'conditional strategies', which allow us to reformulate an asymmetric game as a symmetric game for which we already have an ESS solution concept, and then translate the result back to the asymmetric situation.

This is how it works: suppose that (1) every contest is between a pair of individuals one of which is in role X (e.g., 'larger', 'older', 'stronger') and the other in role Y (e.g., 'smaller', 'younger', 'weaker'); (2) both players know for sure which role they occupy; (3) both players have the same set of conditional strategies from which they can choose.

A conditional strategy is of the form $(x, y) \in X \times Y$, meaning "if in role X, play strategy $x \in X$; if in role Y, play strategy $y \in Y$". The sets X and Y need not be the same. Although a player may find itself in a given role, a conditional strategy is role-independent: only its expression depends on the given role. In other words, the game has become symmetric again, and so we can apply the same old notion of evolutionary stability to conditional strategies.

25. Let $z = (x, y)$ and $z' = (x', y')$ be two conditional strategies. The expected payoff to z' against z is

$$E(z', z) \stackrel{\text{def}}{=} \frac{1}{2}\pi_1(x', y) + \frac{1}{2}\pi_2(x, y')$$

Applying the definition of an ESS, we have that z is an ESS if for every $z' \neq z$ either

$$E(z', z) < E(z, z)$$

or

$$E(z', z) = E(z, z) \quad \& \quad E(z', z') < E(z, z')$$

Writing out these conditions for the payoff functions π_1 and π_2 we get:

Definition. The conditional strategy $(x, y) \in X \times Y$ is an ESS if for all $(x', y') \neq (x, y)$ either

$$\pi_1(x', y) + \pi_2(x, y') < \pi_1(x, y) + \pi_2(x, y)$$

or

$$\begin{aligned} &\pi_1(x', y) + \pi_2(x, y') = \pi_1(x, y) + \pi_2(x, y) \\ \& \quad &\\ &\pi_1(x', y') + \pi_2(x', y') < \pi_1(x, y') + \pi_2(x', y) \end{aligned}$$

26. I think the following proposition and its corollary are actually quite neat:

Proposition. The conditional strategy $(x, y) \in X \times Y$ is an ESS if and only if it is a *strict* Nash equilibrium.

Proof. By definition, (x, y) is a *strict* Nash equilibrium if

$$\begin{aligned} \pi_1(x', y) &< \pi_1(x, y) \quad \forall x' \neq x \\ \pi_2(x, y') &< \pi_2(x, y) \quad \forall y' \neq y \end{aligned}$$

To prove the implication in one direction, suppose (x, y) is a strict Nash equilibrium, then

$$\pi_1(x', y) + \pi_2(x, y') < \pi_1(x, y) + \pi_2(x, y)$$

whenever $x' \neq x$ or $y' \neq y$, i.e., $(x', y') \neq (x, y)$, and so (x, y) is an ESS.

To prove the implication in the other direction, suppose that (x, y) is an ESS. First we show that (x, y) is a Nash equilibrium. From the definition we know that

$$\pi_1(x', y) + \pi_2(x, y') \leq \pi_1(x, y) + \pi_2(x, y) \quad \forall (x', y')$$

For $x' \neq x$ and $y' = y$ this gives

$$\pi_1(x', y) \leq \pi_1(x, y) \quad \forall x'$$

and likewise, for $x' = x$ and $y' \neq y$,

$$\pi_2(x, y') \leq \pi_2(x, y) \quad \forall y'$$

which shows that (x, y) is a Nash equilibrium.

Now we show that (x, y) is a *strict* Nash equilibrium. To reach a contradiction, suppose that this were not true. Then

$$\exists\, x'' \neq x : \quad \pi_1(x'', y) = \pi_1(x, y)$$

or

$$\exists\, y'' \neq y : \quad \pi_2(x, y'') = \pi_2(x, y)$$

Without loss of generality we assume the latter. Then for $(x', y') = (x, y'')$ we have

$$\pi_1(x', y) + \pi_2(x, y') = \pi_1(x, y) + \pi_2(x, y'') = \pi_1(x, y) + \pi_2(x, y)$$

and so the first ESS condition fails. Since by assumption (x, y) is an ESS, the second ESS condition then must hold, i.e.,

$$\pi_1(x', y') + \pi_2(x', y') < \pi_1(x, y') + \pi_2(x', y)$$

which for $(x', y') = (x, y'')$ becomes

$$\pi_1(x, y'') + \pi_2(x, y'') < \pi_1(x, y'') + \pi_2(x, y) = \pi_1(x, y'') + \pi_2(x, y'')$$

and so

$$\pi_2(x, y'') < \pi_2(x, y'')$$

which is a contradiction that proves that (x, y) is in fact a *strict* Nash equilibrium. This completes the proof of the proposition. □

Corollary. An evolutionarily stable conditional strategy $(x, y) \in X \times Y$ always consists of pure strategies.

Proof. A Nash equilibrium in which one of the strategies is a mixed strategy is, by the Bishop-Cannings theorem, not a *strict* Nash equilibrium. □

I think this is quite neat, because finding a candidate mixed ESS and proving that it is an ESS indeed, is not always trivial for games with more than two pure strategies. However, in asymmetric games the ESS *never* is a mixed ESS.

Moreover, the nature of the asymmetry nor the degree of asymmetry do not matter at all as long as both players recognize the difference. The assignment of roles is more like a pre-game contract than a signal: there may be no correlation between the asymmetry and the actual strength of the contestants, for example.

27. Consider the Hawk-Dove game where the resource is a territory and where the roles are defined as territory 'owner' (row-player) and 'intruder' (column-player). The strategy set consists of four conditional strategies: {(H,H), (H,D), (D,H), (D,D)} where the first of each pair is adopted in the role of 'owner' and the second in the role of 'intruder'.

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	$\frac{1}{2}R$, $\frac{1}{2}R$

From the payoff matrix we immediately see that:

- If $R > C$, then (H,H) is a strict Nash equilibrium and therefore is an ESS.
- If $R = C$, then (H,H) is a Nash equilibrium but not a strict Nash equilibrium, and therefore is not an ESS. In fact, there is no ESS.
- If $R < C$, then (H,D) and (D,H) are both strict Nash equilibria, and therefore both are ESS.

The conditional strategy (H,D), which prescribes that 'if you're the owner of a territory, play Hawk; if you're an intruder, play Dove', is called the Bourgois strategy. The opposite, (D,H), which prescribes that 'if you're the owner of a territory, play Dove; if you're an intruder, play Hawk', is called the Paradoxical strategy, because it goes contrary to our instincts. Both the Bourgois and the Paradoxical strategies are evolutionarily stable whenever $R < C$.

As an example of the Paradoxical strategy, John Maynard Smith (1982) mentions the social spider *Oecibus civitas*: these spiders live in groups, but each has its own web and refuge hole. If a spider is driven from its refuge hole, it darts off and enters another web. The owner of that web then darts off and enters another web. So, an initial intrusion causes an avalanche of displacements. The Bourgois strategy, however, is by far the more common of the two ESSes.

28. Suppose that in the Hawk-Dove game the asymmetry is due to differences in strength rather than ownership. Then we find the same ESS, including the Paradoxical strategy in which the stronger player immediately surrenders the resource to the weaker player without a fight: this is absurd!

Let's assume that the stronger player has a probability p (close to one) of winning an escalated fight against a weaker opponent. The payoff matrix then becomes:

	H	D
H	$pR-(1-p)C \ , \ (1-p)R-pC$	$R \ , \ 0$
D	$0 \ , \ R$	$\frac{1}{2}R \ , \ \frac{1}{2}R$

We immediately see that:

- (H,H) is an ESS $\iff pR > (1-p)C$ and $(1-p)R > pC$.
- (H,D) is an ESS $\iff (1-p)R < pC$.
- (D,H) is an ESS $\iff pR < (1-p)C$.
- (D,D) is never an ESS.

This is equivalent to:

- (H,H) is an ESS $\iff C/(R+C) < p < R/(R+C)$.
- (H,D) is an ESS $\iff p > R/(R+C)$.
- (D,H) is an ESS $\iff p < C/(R+C)$.
- (D,D) is never an ESS.

Graphically, this can be represented by the following two diagrams:

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                 (D,H)             (H,H)             (H,D)
Case R>C:   |---------------|---------------|---------------|
            0           C/(R+C)          R/(R+C)            1
                                   p -->
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and

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                (D,H)           (D,H)&(H,D)          (H,D)
Case R<C:    |---------------|---------------|---------------|
             0           R/(R+C)         C/(R+C)             1
                                   p -->
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It is immediately clear that for p sufficiently close to one, the strategy (H,D), i.e., 'choose Hawk if you are stronger than your opponent; choose Dove if you are weaker', is the only ESS.

29. Differences in gender or ownership are usually quite clear to both players. Differences in strength, size or age, however, are may be less obvious and may require some pre-game assessment. This assessment may take the form of a display contest and may be quite costly. The question arrises whether it is worth the trouble to assign roles at all, or whether it is more profitable to play an unconditional strategy. The following model is not very good (for reasons given later), but it may give some idea how to tackle this question.

Consider the strategies Hawk, Dove and Assessor. The Assessor chooses Hawk if in one role (whichever role this is) and Dove in the other role. There is a cost of assessment, however, which we denote by γ. We also assume that after assessment, the Assessor finds itself in either role with probability one-half.

Let $x = (p, 1-p)$ denote a mixed strategy for choosing Hawk or Dove, and let A denote the Assessor strategy. Then

$$\begin{aligned}
\pi_1(x, A) &= \tfrac{1}{2}\pi_1(x, H) + \tfrac{1}{2}\pi_1(x, D) \\
&= \tfrac{1}{2}\Big(p \cdot \tfrac{R-C}{2} + (1-p) \cdot 0\Big) + \tfrac{1}{2}\Big(p \cdot R + (1-p) \cdot \tfrac{R}{2}\Big) \\
&= \tfrac{R}{4} + \tfrac{1}{2}\Big(R - \tfrac{C}{2}\Big)p \\
\pi_1(A, A) &= \tfrac{1}{2}\pi_1(H, D) + \tfrac{1}{2}\pi_1(D, H) - \gamma \\
&= \tfrac{1}{2}R - \tfrac{1}{2} \cdot 0 - \gamma \\
&= \tfrac{1}{2}R - \gamma
\end{aligned}$$

The Assessor strategy is evolutionarily stable if and only if $\pi_1(x, A) < \pi_1(A, A)$ for all x, i.e.,

$$(*) \quad \frac{R}{4} + \frac{1}{2}\Big(R - \frac{C}{2}\Big)p < \frac{1}{2}R - \gamma \quad \forall x$$

We distinguish the following three cases:

Case: $0 < R \leq \frac{1}{2}C$
The left hand side of $(*)$ has a maximum for $p = 0$. Substitution of $p = 0$ into $(*)$ shows that the Assessor strategy is an ESS if and only if $0 < \gamma < R/4$.

Case: $\frac{1}{2}C < R < C$
The left hand side of $(*)$ has a maximum for $p = 1$. Substitution of $p = 1$ into $(*)$ shows that the Assessor strategy is an ESS if and only if $0 < \gamma < (C - R)/4$.

Case: $C < R$
The left hand side of $(*)$ has a maximum for $p = 1$, but substitution of $p = 1$ into $(*)$ requires that $\gamma < 0$, which cannot be satisfied. Instead, the pure strategy Hawk is evolutionarily stable.

We conclude that assessing asymmetry in the Hawk-Dove game and playing the conditional strategy (H,D) is evolutionarily stable provided the cost of the assessment is not too high and the value of the resource does not exceed the cost of injury.

The problem with the above model is the probability of finding oneself in either role is hardly ever one-half. Consider for example asymmetry in strength: if the Assessor is one of the stronger individuals in the population, then the probability of meeting a weaker opponent is definitely greater than one-half. The probability of having a stronger or weaker opponent depends on the Assessor's rank in the population as a whole.

This is easily corrected in the model if we assume that the Assessor knows his own rank. The interesting thing is that whether the Assessor strategy is evolutionarily stable, or not, now also depends on the Assessor's rank in the population as a whole. To work this out is left as an exercise.