SI-model

General

- **Individual-level processes**

$$1 \xrightarrow{\lambda} S \quad (\text{constant birth})$$
$$S \xrightarrow{\mu} \dagger \quad (\text{natural death})$$
$$I \xrightarrow{\mu} \dagger \quad (\text{natural death})$$
$$I \xrightarrow{\mu} \dagger \quad (\text{disease induced death})$$
$$S + I \xrightarrow{\beta} I + I \quad (\text{transmission})$$

- **Population equations**

$$\frac{dS}{dt} = \lambda - \mu S - \beta S I$$

$$\frac{dI}{dt} = \beta S I - (\alpha + \mu) I$$

- **Positive equilibrium for constant $(\alpha, \beta) = (\overline{\alpha}, \overline{\beta})$**

$$\overline{S} = \frac{\overline{\alpha} + \mu}{\overline{\beta}}$$

$$\overline{I} = \frac{\lambda}{\overline{\alpha} + \mu} - \frac{\mu}{\overline{\beta}}$$

Assume that $\frac{\lambda}{\overline{\alpha} + \mu} > \frac{\mu}{\overline{\beta}}$ so that $\overline{I} > 0$

- **Stability**

$$A := \begin{pmatrix} -\frac{\beta \lambda}{\alpha+\mu} & -\alpha - \mu \\ -\mu + \frac{\beta \lambda}{\alpha+\mu} & 0 \end{pmatrix} \quad (\text{Jacobi matrix})$$

Note that Tr[A] < 0 and Det[A] > 0, and so $(\overline{S}, \overline{I})$ is stable

- **Linearization for small fluctuations in (α, β)**

$$\frac{du}{dt} = A\,u + B\,v$$

where

$$u := \begin{pmatrix} S - \overline{S} \\ I - \overline{I} \end{pmatrix}, \quad v := \begin{pmatrix} \alpha - \overline{\alpha} \\ \beta - \overline{\beta} \end{pmatrix}$$

$$B := \begin{pmatrix} 0 & \frac{-\overline{\beta}\,\lambda + \mu\,(\overline{\alpha}+\mu)}{\overline{\beta}^2} \\ \frac{\mu}{\overline{\beta}} - \frac{\lambda}{\overline{\alpha}+\mu} & \frac{\overline{\beta}\,\lambda - \mu\,(\overline{\alpha}+\mu)}{\overline{\beta}^2} \end{pmatrix}$$

- **Fourier transform and transfer function**

$$i\,\omega\,\tilde{u}[\omega] = A\,\tilde{u}[\omega] + B\,\tilde{v}[\omega]$$

$$T[\omega] = (i\,\omega\,\mathrm{Id} - A)^{-1}\,B$$

$$= \begin{pmatrix} \frac{(\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta})}{\overline{\beta}\,(\omega^2\,(\mu+\overline{\alpha}) - i\,\lambda\,\omega\,\overline{\beta} + (\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta}))} & -\frac{(\mu+\overline{\alpha})\,(\mu + i\,\omega + \overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta})}{\overline{\beta}^2\,(\omega^2\,(\mu+\overline{\alpha}) - i\,\lambda\,\omega\,\overline{\beta} + (\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta}))} \\ \frac{(-\mu\,(\mu+\overline{\alpha}) + \lambda\,\overline{\beta})\,(i\,\omega\,(\mu+\overline{\alpha}) + \lambda\,\overline{\beta})}{(\mu+\overline{\alpha})\,\overline{\beta}\,(\omega^2\,(\mu+\overline{\alpha}) - i\,\lambda\,\omega\,\overline{\beta} + (\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta}))} & \frac{(\mu + i\,\omega)\,(\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta})}{\overline{\beta}^2\,(\omega^2\,(\mu+\overline{\alpha}) - i\,\lambda\,\omega\,\overline{\beta} + (\mu+\overline{\alpha})\,(\mu\,(\mu+\overline{\alpha}) - \lambda\,\overline{\beta}))} \end{pmatrix}$$

Example

- **Parameter values**

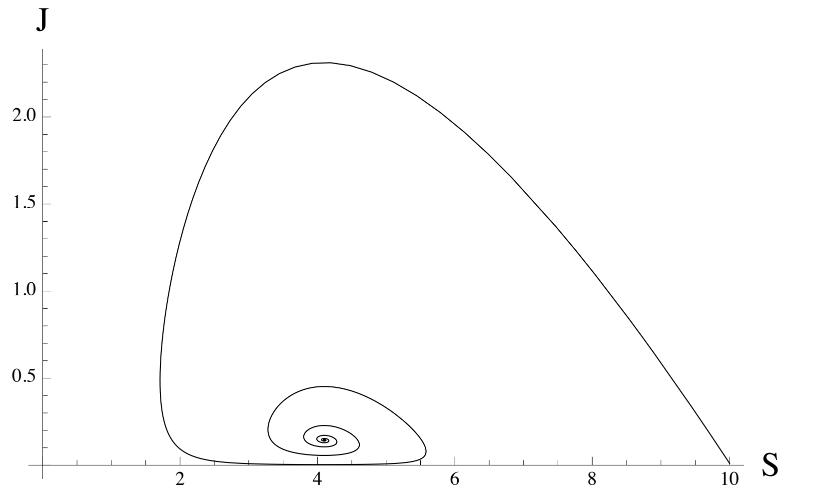
$\lambda = 1,\ \mu = .1,\ \overline{\alpha} = 4,\ \overline{\beta} = 1$

- **Equilibrium and eigenvalues Jacobi matrix**

$\overline{S} = 4.10,\ \overline{I} = 0.14$

Eigenvalues $-0.12 + 0.76\,i$ and $-0.12 - 0.76\,i$

- **Orbit for constant $(\alpha, \beta) = (\overline{\alpha}, \overline{\beta})$**



- **Orbit for fluctuating α and constant $\beta = \overline{\beta}$**

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α = α (1 + ∈ Sin[ω t]) for ∈ = .04 and ω = .75
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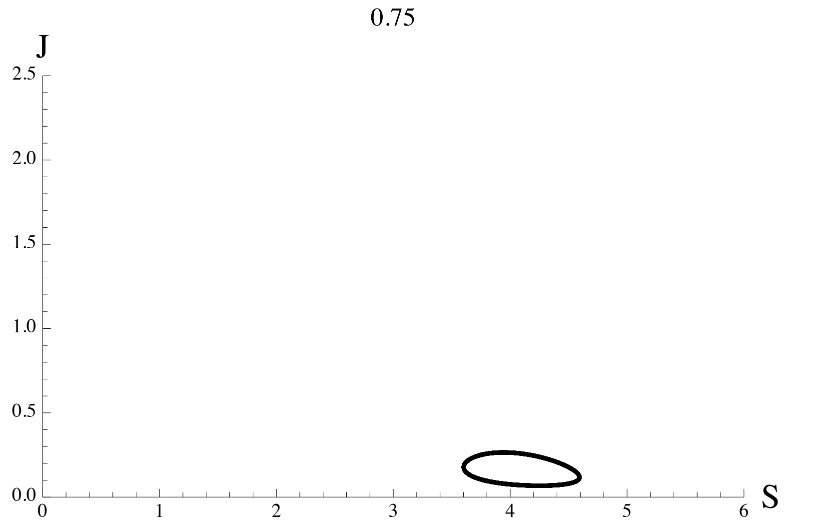
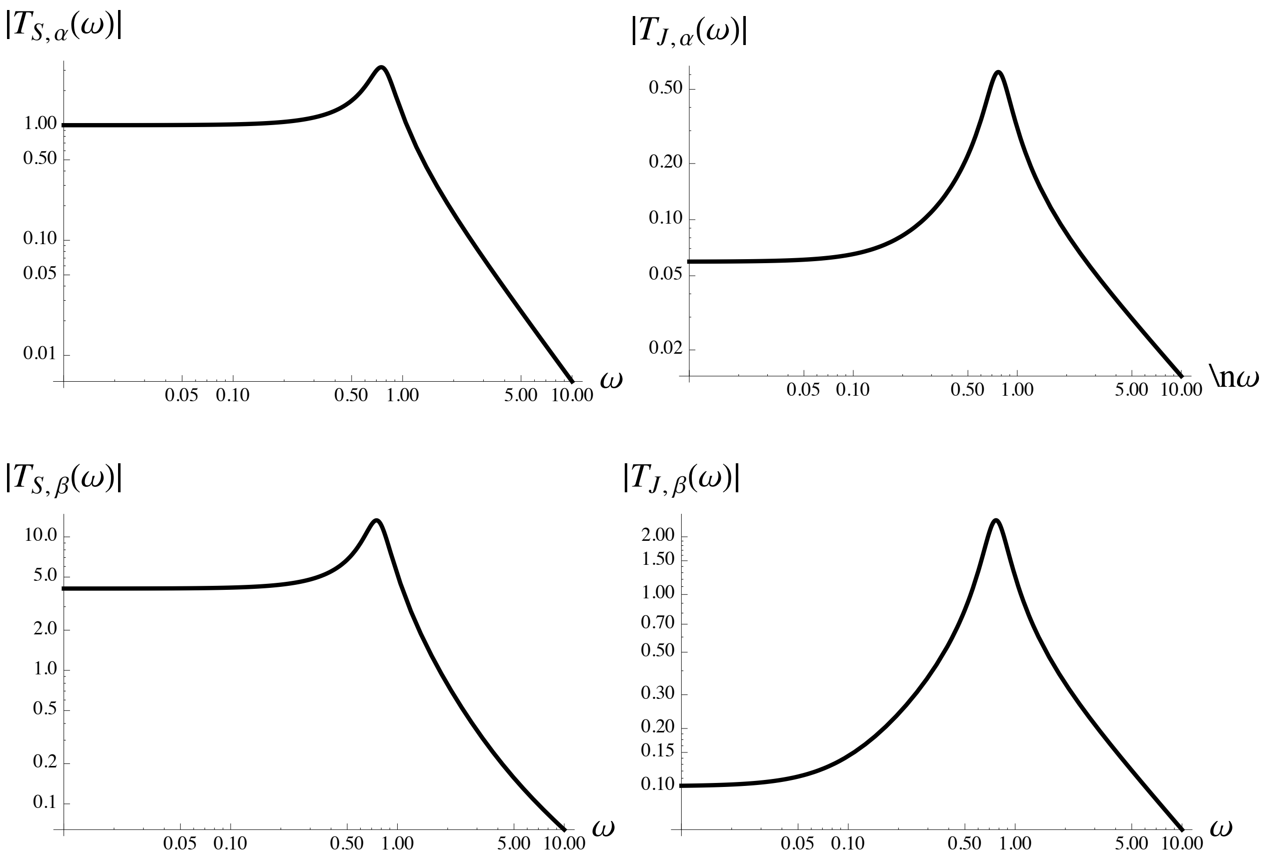


- **Frequency response curves**



- **Resonance frequency**

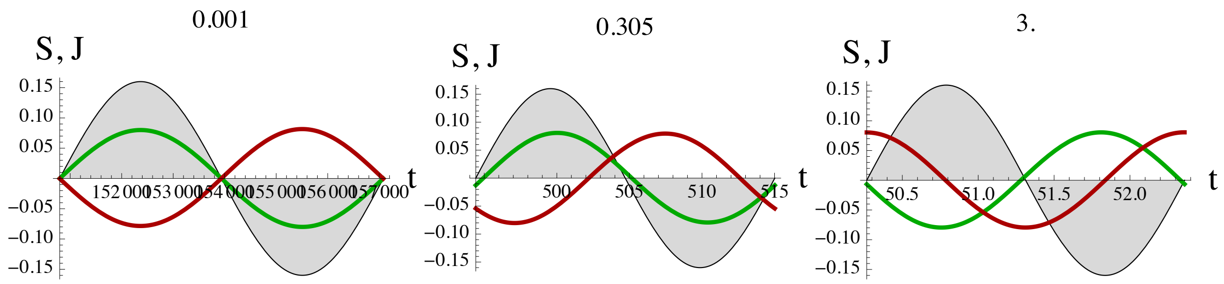
resonance frequencies $\begin{pmatrix} \omega_{S,\alpha} & \omega_{I,\alpha} \\ \omega_{S,\beta} & \omega_{I,\beta} \end{pmatrix} = \begin{pmatrix} 0.748502 & 0.74916 \\ 0.766335 & 0.767792 \end{pmatrix}$

Eigen frequency (from eigenvalues) is 0.758372

GENERALLY, THE RESONANCE FREQUENCIES AND THE EIGENFREQUENCY ARE NOT THE SAME,
BUT TYPICALLY THEY ARE CLOSE TO ONE ANOTHER.

- **Phaseshift for variations in α for $\omega = 0.001$, $\omega = 0.305$, $\omega = 3.0$**

α (gray), S (green), I (red)



- **Phaseshift for variations in β for $\omega = 0.001$, $\omega = 0.305$, $\omega = 3.0$**

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α (gray), S (green), I (red)
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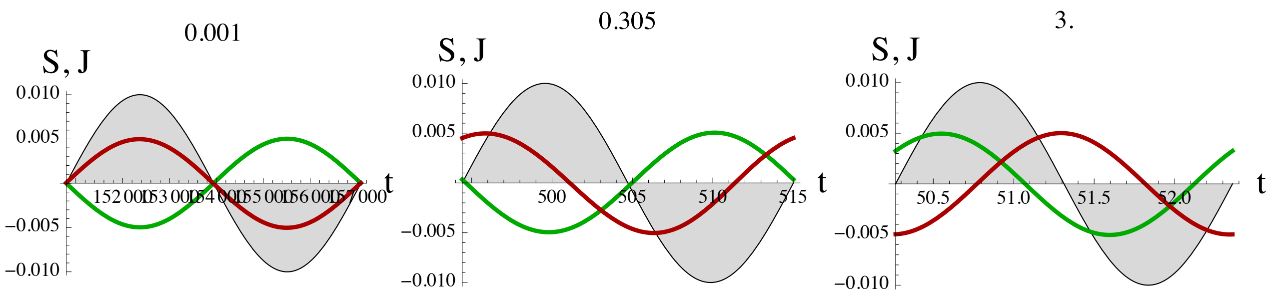

- **Phase response curves**

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S (green),
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