STOCHASTIC POPULATION MODELS

EXERCISES 15-17

15.
Consider a situation in which individuals can die but not reproduce and where the population is maintained by immigration. Assume that death and immigration are independent Poisson processes with a *per capita* death rate equal to δ and a total immigration rate equal to $\alpha\Omega^p$ where Ω is the system size and $0 \leq p \leq 1$. Let further $N(t)$ be the number of individuals present at time $t \geq 0$, and let $P_n(t)$ denote the probability that $N(t) = n$ for any $n \in \{0, 1, 2, \dots\}$.

(a) Give a system of differential equations for the probabilities P_n.

(b) Show that the stationary distribution is the Poisson distribution. Give the mean and variance of the stationary distribution.

(c) Give a possible interpretation of the parameter $p = 0, \frac{1}{2}, 1$.

16.
Derive the quasi-stationary distribution of the linear birth-death process with *per capita* birth rate β and *per capita* death rate δ with $0 < \beta < \delta$ (see section 8.8 of the lecture notes).

17.
Derive the quasi-stationary distribution of a birth-death process with a constant *population level* birth rate $B_n = B > 0$ for all $n \geq 0$ and a constant *population level* death rate $D_n = D > 0$ for all $n \geq 1$ and $D_0 = 0$. Are there conditions on the B and D for which there does not exist a quasi-stationary distribution? What real-life situation could this model represent?