STOCHASTIC POPULATION MODELS

EXERCISES 10-12

10.
Solve the stochastic differential equation

$$\mathrm{d}X(t) = X(t)\left(\frac{1}{2}b^2 - a\log X(t)\right)\mathrm{d}t + bX(t)\,\mathrm{d}W(t) \quad \text{(Ito)}$$

with $X(0) = x_0$ a.s. and $a, b > 0$, using the substitution $Y(t) = \log X(t)$.

11.
Let $X(t)$ be a solution of the stochastic differential equation

$$\mathrm{d}X = g(X,t)\,\mathrm{d}W \quad \text{(Ito)}.$$

Calculate the time-derivative of $Y(t) = h(X(t), W(t))$.

12.
Consider the stationary stochastic processes $X(t)$ and $Y(t)$, both with zero mean, and $Z(t) = aX(t) + bY(t)$, and show that

(a) $C_{Z,Z}(\tau) = a^2 C_{X,X}(\tau) + ab\,C_{X,Y}(\tau) + ab\,C_{Y,X}(\tau) + b^2 C_{Y,Y}(\tau)$.

(b) $C_{X,Y}(\tau) = C_{Y,X}(-\tau)$.

(c) $C_{\dot{X},Y}(\tau) = C'_{X,Y}(\tau)$

(d) $C_{X,\dot{Y}}(\tau) = -C'_{X,Y}(\tau)$

(e) $C_{\dot{X},\dot{Y}}(\tau) = -C''_{X,Y}(\tau)$

where $\dot{X}$ and $\dot{Y}$ are the time-derivatives of $X(t)$ and $Y(t)$, and where $C'_{X,Y}(\tau)$ and $C''_{X,Y}(\tau)$ are, respectively, the first- and second-order derivatives of the cross-covariance function $C_{X,Y}(\tau)$.