Delayed logistic with fixed maturation time τ

Model with constant parameters

- **Individual states**

X : adult individual
Y : juvenile individual

- **Individual-level processes**

$$
\begin{array}{lll}
X \xrightarrow{\beta} X + Y & \text{(birth)} \\
X + X \xrightarrow{\gamma} X + \dagger & \text{(competition)} \\
X \xrightarrow{\delta} \dagger & \text{(death)} \\
Y \xrightarrow{\alpha} \dagger & \text{(death)} \\
Y \overset{\tau}{\Longrightarrow} X & \text{(maturation)}
\end{array}
$$

- **Population equation**

$$\frac{dx}{dt} = \beta\, x_\tau\, e^{-\alpha\tau} - \delta\, x - \frac{\gamma}{2}\, x^2$$

where $x_\tau[t] := x[t-\tau]$

- **Positive equilibrium**

$$\bar{x} = \frac{2}{\gamma}\left(\beta\, e^{-\alpha\tau} - \delta\right) > 0$$

- **Linearization about the equilibrium**

$$\frac{du}{dt} = a\, u + b\, u_\tau$$

where $u := x - \bar{x}$ and $u_\tau := x_\tau - \bar{x}$ and

$a = \delta - 2\,\beta\, e^{-\alpha\tau} < 0$ and $b = \beta\, e^{-\alpha\tau} > 0$

Note that γ has disappeared from the equation

- **Local stability**

Note that $b\,\tau = \frac{1}{2}\,(\delta\,\tau - a\,\tau)$, so that only the pink region in the figure below can be realized. In other words, $\bar{x}$ is always stable and overdamped;

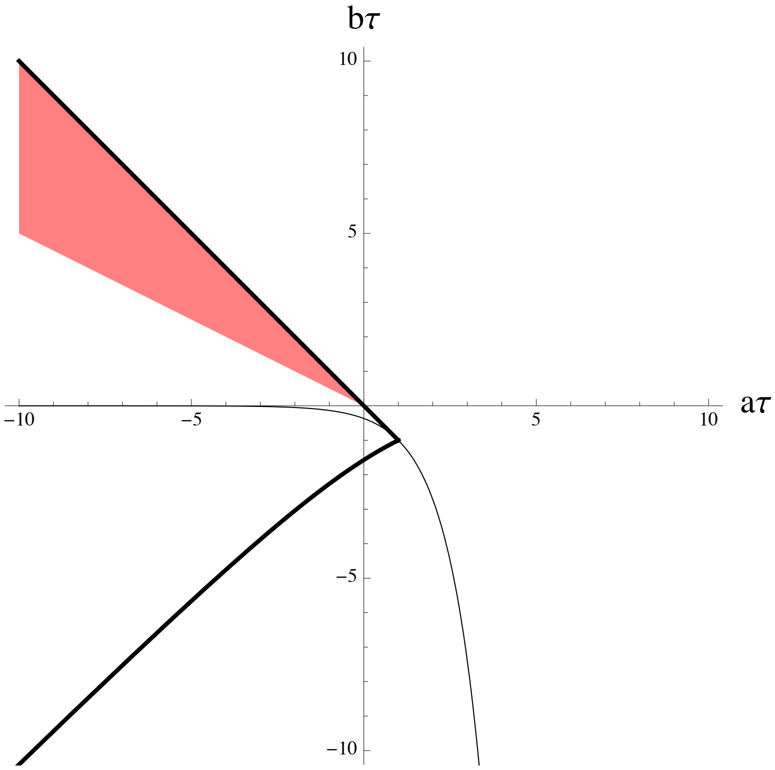
```
StabilityPlot =
 Show[
  ParametricPlot[{ντ/Tan[ντ], -ντ/Sin[ντ]}, {ντ, 0, π}, PlotStyle → {Black, Thick}] // Quiet,

  Graphics[{Pink, Polygon[{{0, 0}, {-10, 10}, {-10, 5}}]}],

  Plot[-aτ, {aτ, -10, 1}, PlotStyle → {Black, Thick}],

  Plot[-e^(aτ-1), {aτ, -10, 10}, PlotStyle → Black],

  PlotRange → {{-10, 10}, {-10, 10}}, AxesOrigin → {0, 0},
  AxesLabel → {"aτ", "bτ"}, AspectRatio → 1, ImageSize → Medium]
```



Model with fluctuating birthrate β

- **Population equation**

$$\frac{dx}{dt} = \beta_\tau\, x_\tau\, e^{-\alpha\,\tau} - \delta\, x - \frac{\gamma}{2}\, x^2$$

where $\beta_\tau[t] := \beta[t - \tau]$ (fixed delay in driver)

- **Linearization for small variations of β around the constant $\overline{\beta}$**

$$\frac{du}{dt} = a\,u + b\,u_\tau + c\,v_\tau$$

where $u := x - \overline{x}$ and $v := \beta - \overline{\beta}$

$a = \delta - 2\,\overline{\beta}\,e^{-\alpha\,\tau}$ and $b = \overline{\beta}\,e^{-\alpha\,\tau}$ and $c = \overline{x}\,e^{-\alpha\,\tau}$

- **Transfer function**

$$i\,\omega\,\tilde{u} = a\,\tilde{u} + b\,e^{-i\,\omega\,\tau}\,\tilde{u} + c\,e^{-i\,\omega\,\tau}\,\tilde{v}$$

and hence for the transfer function we find

$$T[\omega] = \frac{c\,e^{-i\,\omega\,\tau}}{i\,\omega - a - b\,e^{-i\,\omega\,\tau}}$$

- **Numerical example**

```
α = 1; β̄ = 10; γ = 1; δ = 1; τ = 1;
```

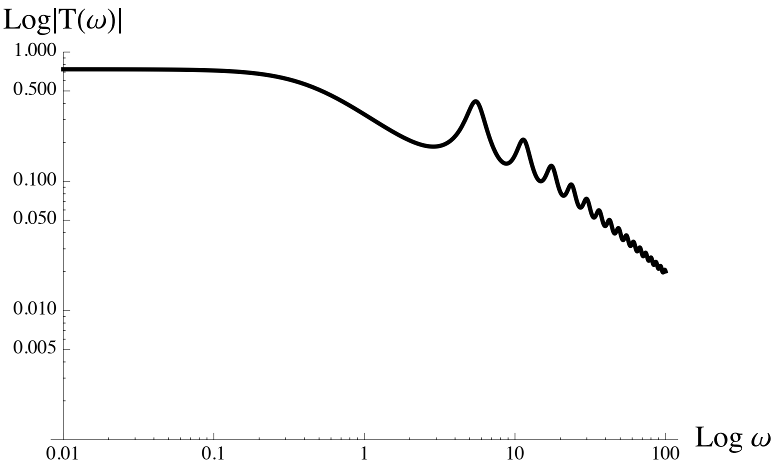
$$\bar{x} = \frac{2}{\gamma}\left(\bar{\beta}\, e^{-\alpha\tau} - \delta\right) \text{ // N}$$

```
5.35759
```

$$a = \delta - 2\bar{\beta}\, e^{-\alpha\tau};\ b = \bar{\beta}\, e^{-\alpha\tau};\ c = \bar{x}\, e^{-\alpha\tau};$$

$$T[\omega\_] := \frac{c\, e^{-i\omega\tau}}{i\omega - a - b\, e^{-i\omega\tau}};$$

```
Show[
 LogLogPlot[Abs[T[ω]], {ω, .01, 100},
  PlotPoints → 100, PlotStyle → {Black, Thick}, PlotRange → {.001, 1}],
 AxesLabel → {"Log ω", "Log|T(ω)|"}, ImageSize → Medium]
```



Model with fluctuating juvenile death rate α

- **Population equation**

$$\frac{dx}{dt} = \beta\, x_T\, e^{-\alpha_\psi \tau} - \delta\, x - \frac{\gamma}{2} x^2$$

where $\alpha_\psi[t] := \int_{-\infty}^{+\infty} \alpha[t-s]\, \psi[s]\, ds$ (distributed delay in driver) and

where $\psi[t] := \begin{cases} \tau^{-1} & \text{for } 0 \le t \le \tau \\ 0 & \text{otherwise} \end{cases}$

Alternatively we can write $\alpha_\psi = \alpha * \psi$ (convolution)

- **Linearization for small variations of α around the constant $\overline{\alpha}$**

$$\frac{du}{dt} = a\,u + b\,u_\tau + c\,v_\psi$$

where $u := x - \overline{x}$ and $v := \alpha - \overline{\alpha}$ and hence $v_\psi = v \star \psi$

$a = \delta - 2\,\beta\,e^{-\overline{\alpha}\,\tau}$ and $b = \beta\,e^{-\overline{\alpha}\,\tau}$ and $c = -\beta\,\tau\,\overline{x}\;e^{-\overline{\alpha}\,\tau}$

- **Transfer function**

$$i\,\omega\,\tilde{u} = a\,\tilde{u} + b\,e^{-i\,\omega\,\tau}\,\tilde{u} + c\,e^{-i\,\omega\,\tau}\,\tilde{v}\,\tilde{\psi}$$

where from the definition of the Fourier transform we find

$$\tilde{\psi}[\omega] = \frac{1 - e^{-i\,\omega\,\tau}}{i\,\omega\,\tau}$$

Hence for the transfer function we find

$$T[\omega] = \frac{c\,e^{-i\,\omega\,\tau}\left(1 - e^{-i\,\omega\,\tau}\right)}{i\,\omega\,\tau\left(i\,\omega - a - b\,e^{-i\,\omega\,\tau}\right)}$$

Note that $T[\omega] = 0$ whenever $e^{-i\,\omega\,\tau} = 1$, i.e., for $\omega\,\tau = 2\,\pi\,k$, $k \in \mathbb{N}$
in which case the period of the driver is exactly an integer multiple of the delay τ

■ Numerical example

$\bar{\alpha}$ = 1; β = 10; γ = 1; δ = 1; τ = 1;

$$\bar{x} = \frac{2}{\gamma}\left(\beta\, e^{-\bar{\alpha}\tau} - \delta\right) \text{ // N}$$

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$a = \delta - 2\beta\, e^{-\bar{\alpha}\tau}$; $b = \beta\, e^{-\bar{\alpha}\tau}$; $c = -\beta\,\tau\,\bar{x}\, e^{-\bar{\alpha}\tau}$;

$$T[\omega\_] := \frac{c\, e^{-i\omega\tau}\left(1 - e^{-i\omega\tau}\right)}{i\,\omega\,\tau\left(i\,\omega - a - b\, e^{-i\omega\tau}\right)};$$

```
Show[
 LogLogPlot[Abs[T[ω]], {ω, .01, 100},
  PlotStyle → {Black}, PlotRange → {.001, 10}, PlotPoints → 100],
 AxesLabel → {"Log  ω", "Log|T(ω)|"}, ImageSize → Medium]
```