**UH Stochastic analysis II, Spring 2017, Exercise-7 (11.4.2017)**

1. Consider a gamma process $X(t)$ with independent homogeneous increments and Lévy measure

$$\nu(dx) = \beta e^{-\alpha x} x^{-1} dx$$

and consider the martingale $\widetilde{X}(t) = X(t) - tE(X_1) = X(t) - t\beta/\alpha$.

   (a) Compute its quadratic variation $[\widetilde{X}]_t$ and predictable variation $\langle \widetilde{X} \rangle_t$.

   (b) Compute the quadratic variation and predictable variations the martingale

$$I(t) := \int_0^t \widetilde{X}(s-) d\widetilde{X}(s)$$

   where the integral is defined pathwise in Lebesgue Stieltjes sense.

   (c) Show that $I(t)$ is also an Ito integral with respect to the $L^2(P)$-martingale $\widetilde{X}(t)$.

2. (Lenglart's inequality)

   Let $X_t(\omega) \geq 0$ with $X_0 = 0$, and $A_t(\omega) \geq 0$ continuous processes adapted with respect to $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, and assume that $A_t$ is non-decreasing such that for all **bounded** stopping times $\tau(\omega)$

$$E(X_\tau) \leq E(A_\tau) \tag{0.1}$$

   We introduce the running maximum $X_t^*(\omega) = \max_{0 \leq s \leq t} X_s(\omega)$ which is a non-decrasing process and therefore has finite variation on compact intervals.

   Prove the following inequalities for **all** $\mathbb{F}$-stopping times $\tau$, (also unbounded ): $\forall \varepsilon, \delta > 0$

   **a**) $\quad P(X_\tau^* > \varepsilon) \leq \dfrac{E(A_\tau)}{\varepsilon}$

   **b**) $\quad P(X_\tau^* > \varepsilon, A_\tau \leq \delta) \leq \dfrac{E(A_\tau \wedge \delta)}{\varepsilon}$

   **c**) $\quad P(X_\tau^* > \varepsilon) \leq \dfrac{E(A_\tau \wedge \delta)}{\varepsilon} + P(A_\tau > \delta)$

Hint: First assume that $\tau$ is a bounded stopping time, then you can use monotone convergence for unbounded stopping times.

Define

$$\sigma(\omega) = \inf\{t : X_t(\omega) > \varepsilon\}$$

and note that

$$\{X^*_\tau > \varepsilon\} = \{\ \sigma < \tau\}$$

Use the assumption (0.1) for the stopping time $\sigma \wedge \tau$.

3. Let $M_t$ a continuous $\mathbb{F}$-local martingale. Then $M$ has $\mathbb{F}$-predictable variation $\langle M\rangle_t$ is the non-decreasing process with $\langle M\rangle_0 = 0$ such that

$$N_t = M_t^2 - \langle M\rangle_t$$

is a $\mathbb{F}$-local martingale.( A continuous local martingale is locally bounded and and Doob decomposition applies to the bouned submartingale $M^2_{t\wedge\tau_n}$ with localizing sequence $\tau_n \ldots$).

Show that for any $\mathbb{F}$-stopping time $\tau$

$$P\biggl(\max_{0\leq s\leq\tau(\omega)} |\ M_s(\omega)|\ > \varepsilon\biggr) \leq \frac{E(\delta \wedge \langle M\rangle_\tau)}{\varepsilon^2} + P(\langle M\rangle_\tau > \delta)$$

Hint: use the previous exercise. In order to show that assumption (0.1) is satisfied, take a localizing sequence for $N_t$ and use Fatou lemma.

4. $\{\ M_t^{(n)}(\omega)\}_{n\in\mathbb{N}}$ a sequence of $\mathbb{F}$-local martingales and $\tau$ a $\mathbb{F}$-stopping time. Show that as $n \longrightarrow \infty$

$$\langle M^{(n)}\rangle_\tau \stackrel{P}{\to} 0 \quad \Longrightarrow \quad \sup_{0\leq s\leq\tau} |M_s^{(n)}(\omega)| \stackrel{P}{\to} 0$$

with convergence in probability.

5. Let $(X_i : i \leq \mathbb{N})$ be i.i.d. $\mathbb{R}$-valued random variables with cumulative distribution function $F(t) = P(X_i \leq t)$.

For sample size $n \in \mathbb{N}$, consider the empirical process

$$F^{(n)}(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i(\omega) \leq t) \qquad t \in \mathbb{R}$$

in its own time-continuous filtration $\mathbb{F}^{(n)} = (\mathcal{F}_t^{(n)} : t \in \mathbb{R})$, where $\mathcal{F}_t^{(n)} = \sigma(F^{(n)}(s) : s \leq t)$, $t \in \mathbb{R}$.

(a) Assuming that $t \mapsto F(t)$ is continuous, compute the compensator (dual predictable projection) $A^{(n)}(t)$ of $F^{(n)}(t)$ in the time continuous filtration $\mathbb{F}^{(n)}$. Note that the time parameter runs over the all real line $\mathbb{R}$.

(b) What is the compensator when the cumulative distribution function $F(t)$ is cadlag with jumps ?

(c) Consider the martingale

$$M^{(n)}(t) = F^{(n)}(t) - A^{(n)}(t)$$

show that it is bounded in $L^2(P)$. Compute the quadratic variation $[M^{(n)}]_t$ and the predictable variation $\langle M^{(n)}\rangle_t$

(d) Compute an upper bound for

$$P\big(\sup_{t\in\mathbb{R}} |M^{(n)}(t)| > \varepsilon\big)$$

(e) Note that for $k \leq n$, $F^{(k)}(t)$ and $M^{(k)}(t)$ are adapted to the $\mathbb{F}^{(n)}$-filtration.

Compute the quadratic covariation $[M^{(k)}, M^{(n)}]_t$ and predictable covariation $\langle M^{(k)}, M^{(n)}\rangle_t$

(f) Compute the covariance

$$E\big(F^{(n)}(t)F^{(k)}(s)\big) - E\big(F^{(n)}(t)\big)E\big(F^{(k)}(s)\big)$$

(g) Show that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \big|M^{(n)}(t)\big| = 0$$

in $L^2(P)$.

Hint: see first what happens with $n = 1$. Compute the compensator of the single jump counting process $N^{(i)}(t) := \mathbf{1}(X_i \leq t)$.

Note also that when $F(t)$ is continuous one can use Kurtz theorem.