UH Stochastic analysis II, Spring 2017, Exercise-6 (4.4.2017)

1. Let $\alpha \in (0,2)$ and $X(t)$ a one-dimensional Lévy process with $b = \sigma = 0$ (without drift and Brownian component) with Lévy measure

$$\nu(dx) = c|x|^{-1-\alpha} dx$$

   (a) Check that this Lévy measure satisy the integrability condition of Lévy Kintchine theorem which guarantees that such Lévy process exists.

   (b) Show that $X(t)$ is α^{-1}**-self-similar:** for $c > 0$

$$\big(c^{-1/\alpha} X(ct) : t \geq 0\big) \stackrel{law}{=} \big(X(t) : t \geq 0\big)$$

   (c) Compute the characteristic function of $X(t)$:

$$\theta \mapsto E\big(\exp(i\theta X(t))\big)$$

   Hint:

$$e^{iux} - 1 = iu \int_0^x e^{iuy} dy,$$
$$e^{iux} - 1 - iux = iu \int_0^x \big(e^{iuy} - 1\big) dy = -u^2 \int_0^x \int_0^y e^{iur} dr dy = -\frac{u^2}{2} \int_0^x \int_0^x e^{iur} dr dy$$
$$= -\frac{u^2 x}{2} \int_0^x e^{iuy} dy$$

   use Fubini in the Lévy Khintchine formula.

   (d) For which values of α the Lévy process $X(t)$ has finite variation on compacts ?

2. (Frullani integral)

   (a) Show that if $f'(x)$ is continuous and the integral below converges, we have

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \big(f(0) - f(\infty)\big)\big(\log(b) - \log(a)\big)$$

   Hint: use Fubini.

(b) Use Frullani integral to show that $\forall \alpha, \beta > 0$ $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq 0$ we have

$$\big(1 - z/\alpha\big)^{-\beta} = \exp\bigg(\int_0^\infty (e^{zx} - 1)\beta e^{-\alpha x} x^{-1} dx\bigg)$$

3. A a one-dimensional Lévy process with $b = \sigma = 0$ (without drift and Brownian component) with Lévy measure

$$\nu(dx) = \mathbf{1}(x > 0)\beta x^{-1}\exp(-\alpha x)dx$$

where α and $\beta > 0$ is called Gamma process.

(a) Check that $\nu(dx)$ satisfies the integrability assumptions of Lévy Kintchine theorem, which guarantees the existence of such Gamma process.

(b) Check that $X(t)$ has finite first variation although $\nu(\mathbb{R}^+) = \infty$ and it has infinitely many small jumps.

(c) Compute the characteristic function of $X(t)$:

$$\theta \mapsto E\big(\exp(i\theta X(t))\big)$$

(d) Show that $X(t)$ has Gamma distribution, with density

$$p_{\alpha,\beta t}(x) = \frac{\alpha^{\beta t}}{\Gamma(\beta t)} x^{\beta t - 1}\exp(-\alpha x)dx$$

which mean that the increments $X(t) - X(s)$ are gamma distributed.

(e) Compute the moments

$$E(X(t)^n)$$

(f) Compute the covariance

$$E(X(t)X(s)) - E(X(t))E((X(s))$$

(g) Is the gamma process $X(t)$ ρ-selfsimilar of index ρ for some $\rho > 0$?