UH Stochastic analysis II, Spring 2017, Exercise-6 (28.3 2017)

1. Let $t \mapsto X(t)$ a cadlag function with finite variation on compact intervals and let

$$X^c(t) = X(t) - \sum_{s \leq t} \Delta X(s)$$

its continuous part.

We define the *Doleans exponential* $\mathcal{E}(X)_t$ as the solution of the linear equation

$$Z(t) = 1 + \int_0^t Z(s-)X(ds)$$

given by

$$Z(t) = \exp\big(X^c(t)\big) \prod_{s \leq t} \big(1 + \Delta X(s)\big) = \exp\big(X(t)\big) \prod_{s \leq t} \Big\{ \exp(-\Delta(X(s))\big(1 + \Delta X(s)\big) \Big\} =$$

(a) Let $Y(t)$ be another cadlag function with finite variation on compacts. Show that the non-homogeneous stochastic exponential

$$\mathcal{E}^Y(X)_t = \mathcal{E}(X)_t \bigg(Y(0) + \int_0^t \frac{1}{\mathcal{E}(X)_s} Y(ds) \bigg)$$

solves the non-homogeneous linear equation

$$Z(t) = Y(t) + \int_0^t Z(s-)X(ds)$$

Hint: use the integration by parts formula.

(b) Prove Yor formula for cadlag processes X, Y with finite variation on compacts.

$$\mathcal{E}(X)_t \mathcal{E}(Y)_t = \mathcal{E}(X + Y + [X, Y])_t$$

(c) Show that

$$\frac{1}{\mathcal{E}(X)_t} = 1 - \int_0^t \frac{1}{\mathcal{E}(X)_s} X(ds)$$

2. Let $N(t)$ be a counting process with $\Delta N(t) \in \{0,1\}$,

   such that $N(0) = 0$ and $E(N(t)) < \infty$, and let $A(t) = N^p(t)$ be its dual predictable projection (compensator) in the filtration $\mathbb{F}$. Recall that $\Delta A(t) \in [0,1]$.

   Consider the $\mathbb{F}$ martigale $M(t) = N(t) - A(t)$ and show that

$$[M,M]_t - \int_0^t (1 - \Delta A(s)) A(ds)$$

   is a martingale, where

$$[M,M]_t = \sum_{s \le t} \{\Delta M(s)\}^2$$

   is the quadratic variation of M.

3. Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function with $\Lambda(0) = 0$. We associate to Λ the measure $\Lambda(dt)$ with $\Lambda((s,t]) = \Lambda(t) - \Lambda(s)$ for $0 \le s \le t$. Show that there exists a Poisson process $N(t)$ driven by Λ, i.e. $N(0) = 0$ and the increments are Poisson distributed

$$N(t) - N(s) \sim \text{ Poisson}(\Lambda(t) - \Lambda(s)) \text{ for } 0 \le s \le t$$

   independently of the past.

   The compensated Poisson process $\widetilde{N}(t) := N(t) - \Lambda(t)$ is a square integrable martingale in its own filtration. Show that $\widetilde{N}$ is bounded in $L^2(P)$ if and only if $\Lambda(\infty) < \infty$.

   Let

$$\mathcal{H} := L^2(\mathbb{R}_+, d\Lambda) = \left\{ h : \mathbb{R}_+ \to \mathbb{R} \text{ Borel measurable such that } \int_0^\infty h(s)^2 \Lambda(ds) < \infty \right\}$$

   For $h \in \mathcal{H}$, consider the martingale transform defined as Lebesgue Stieltjes integral

$$(h \cdot \widetilde{N})_t = \int_0^t h(s) \widetilde{N}(ds)$$

   (a) Show the isometry between $L^2(\mathbb{R}_+, d\Lambda)$ and $L^2(P)$:

$$\int_0^\infty h(s)^2 \Lambda(ds) = \| h \|_{\mathcal{H}}^2 = \| (h \cdot \widetilde{N})_\infty \|_{L^2(P)}^2 = E\left(\left(\int_0^t h(s) \widetilde{N}(ds) \right)^2 \right)$$

(b) Assume that in the filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$, for each $0 \leq s \leq t$ $(N(t) - N(s)) \perp\!\!\!\perp \mathcal{F}_s$,

$$L^2_{\text{pred}}(\Omega \times \mathbb{R}_+, dP \times d\Lambda) =$$
$$\left\{ h(s,\omega) : \ \mathbb{F}\text{-predictable such that } \int_0^\infty E(h(s)^2)\Lambda(ds) < \infty \right\}$$

(c) Show the isometry between $L^2(\mathbb{R}_+, d\Lambda)$ and $L^2(P)$:

$$\int_0^\infty E(h(s)^2)\Lambda(ds) = \| \, h \, \|^2_{L_2(dP\times d\Lambda)} = \| \, (h \cdot \widetilde{N})_\infty \, \|^2_{L^2(P)} = E\left(\left(\int_0^t h(s)\widetilde{N}(ds) \right)^2 \right)$$

Hint: consider first simple predictable integrands of the form

$$h(s,\omega) = \mathbf{1}(u < s \leq v)\mathbf{1}_A(\omega), \quad A \in \mathcal{F}_u$$

and linear combinations of those.

4. Let $W(t)$ be a Brownian motion and $N(t)$ an independent Poisson process driven by a deterministic non-decreasing continuous function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$, in a filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$, let

$$M(t) = W(t) + N(t) - \Lambda(t)$$

and $\tau_n = \inf\{t : |M(t)| \geq n\}$.

Show that the stopped process $\big(M(t \wedge \tau_n) : t \geq 0\big)$ is a bounded martingale, but the martingale $\big(W(t \wedge \tau_n) : t \geq 0\big)$ is not bounded.