Stochastic analysis, spring 2017, Home Exam

1. Let $(B_t^{(1)}, \dots, B_t^{(n)} : t \geq 0)$ continuous local martingales in the filtration $\mathbb{F}$ with

$$\langle B^{(i)}, B^{(i)} \rangle_t = t,$$
$$\langle B^{(i)}, B^{(j)} \rangle_t = E_P\big(B_t^{(i)} B_t^{(j)}\big) = c_{ij} t, \text{ for } i \neq j, .$$

with $c_{ij} \in [-1, 1]$ constant.

   (a) Each $B_t^{(i)}$ is a Brownian motion. Why ?
   Since B_t is a continuous and square integrable martingale $E(B_t^2) = E(\langle B, B \rangle_t) = E(\langle B \rangle_t) = t$. It is easy to see that at time $t = 0$ this implies $B_0 = 0$ P-almost surely (show it!).

   (b) Assume $B_0^{(i)} = 0$ at time $t = 0$.
   Use inducively Ito formula and Fubini Theorem to compute the joint moment at time t:

$$E_P\big(B_t^{(1)} \dots B_t^{(n)}\big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum\limits_{\text{pairings}} \prod\limits_{pairs\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

   where when n is even, the sum is over all pairings of $1, \dots, n$ into $n/2$ pairs, where the pairs are disjoint and the elements of the pairs are distince. For each pairing we then take the product over the pairs of the pairing.
   Hint: Let's see how this works in practice, for example

$$E_P\big(B_t^{(1)} B_t^{(2)} B_t^{(3)} B_t^{(4)}\big) = (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23})t^2$$

   since we can form disjoint pairs in three possible way, and each pairing contributes with the product of two covariances.
   Another example would be

$$E_P\big((B_t^{(1)})^2 B_t^{(2)} B_t^{(3)}\big) = (c_{11}c_{23} + 2c_{12}c_{13})t^2$$

   and

$$E_P\big((B_t^{(1)})^2 (B_t^{(2)})^2\big) = (c_{11}c_{22} + 2c_{12}^2)t^2$$

   This can be proved by using Ito formula to write the semimartingale decomposition of the product, and then arguing that the martingale part has zero mean.
   Hint: Compute the semimartingale decomposition of the product $B_t^{(1)} \dots B_t^{(n)}$, and show that the local martingale is a true martingale (which therefore has zero expectation).
   This is Wick's formula (in the literature usually the proof is based on the moment generating function).

2. (a) Show that an essentially bounded local martingale (that is for some $K < \infty$, $P(|M_t| < K) = 1 \; \forall t > 0$). is a true martingale.

(b) Let B_t a Brownian motion in the filtration $\mathbb{F}$, and
$Z_t = \exp\big(M_t - t/2\big)$.
Show that Z_t is a continuous martingale which is not uniformly integrable.

3. Let (B_t) be a standard Brownian motion, denote $i = \sqrt{-1}$ as usual. Recall that

$$Z(t,\theta) = \exp\left(i\theta B_t + \frac{1}{2}\theta^2 t\right) =$$
$$\cos(\theta B_t)\exp(\theta^2 t/2) + i\sin(\theta B_t)\exp(\theta^2 t/2) = M_t(\theta) + iN_t(\theta)$$

is a complex valued martingale $\forall\theta \in \mathbb{R}$, that is both real and imaginary parts are martingales.

Compute the brackets $\langle M(\theta), M(\theta)\rangle_t$, $\langle N(\varphi), N(\varphi)\rangle_t$, $\langle M(\theta), N(\varphi)\rangle_t$.

Note that for different angles θ, φ you get different martingales,

$M_t(\theta) + iN_t(\theta)$ and $M_t(\varphi) + iN_t(\varphi)$.

the question is to compute the quadratic cross-covariation also when θ and φ are different.

4. In the setting of exercise 2,

Compute the Ito-Clarck martingale representation of the square integrable random variable

$$X_T = \sin(\theta B_T)\cos(\varphi B_T) = E\big(\sin(\theta B_T)\cos(\varphi B_T)\big) + \int_0^T Y_s dB_s$$

i.e. compute the expectation and find the adapted integrand process Y_s.

Hint. rewrite

$$X_T = cM_T(\theta)N_T(\varphi)$$

with $c = \exp(-(\theta^2 + \varphi^2)T/2)$, and use integration by parts, to find the martingale decomposition of the product $(M_t(\theta)N_t(\varphi))$.

Hint: $X_T(\omega) \in [-1,1]$ is a bounded random variable, simply because $\sin(x)$ and $\cos(x)$ are bounded functions. On a probability space, since $P(\Omega) = 1$, it follows that $L^\infty(\Omega, P) \subseteq L^q(\Omega, P)$ for all powers q, with $\| X \|_{L^q(\infty)} \le \| X \|_{L^\infty(P)}$.

5. Let $X_T = \exp(\theta B_T)B_T^2$, where $\theta \in \mathbb{R}$.

a) Show that $X_T \in L^2(\Omega)$.

Hint: Note that $E(\exp(\theta B_T)) = \exp(\theta^2 T/2) < \infty\ \forall\theta \in \mathbb{R}$.

Note also that the exponential function grows faster than any polynomial, in particular for all $\varepsilon > 0\ \exists C_\varepsilon > 0$ such that

$$x^2 \le C_\varepsilon\big(\exp(\varepsilon x) + \exp(-\varepsilon x)\big) \quad \forall x$$

Using this it is easy to show a.

b) Compute $E(X_T)$.

Hint: the idea is that $\exp(\theta B_T - \theta^2/2)$ corresponds to a change of measure from the measure P to the measure P_θ with likelihood ratio

$$\frac{dP_\theta}{dP}(\omega) = \exp(\theta B_T - \theta^2/2)$$

Under the new measure P_θ, B_T has Gaussian distribution with mean θT and variance T. This follows simply by writing the product of exponentials as exponential of sum and completing the squares, inside the integral

$$\frac{1}{\sqrt{2\pi T}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2T}\right) \exp(\theta x) x^2 dx$$

c) Compute the Ito-Clarck martingale representation of X_T. Hint: use Ito formula and integration by parts.

6. (a) Solve the following Ito SDE

$$\begin{aligned}
a) \quad & X_t = x + \int_0^t \sqrt{1 - X_s^2} dB_s - \frac{1}{2} \int_0^t X_s ds \\
b) \quad & X_t = x + \int_0^t \sqrt{1 + X_s^2} dB_s + \frac{1}{2} \int_0^t X_s ds \\
c) \quad & X_t = x + \int_0^t \sqrt{1 + X_s^2} dB_s + \int_0^t \left(\sqrt{1 + X_s^2} + \frac{1}{2} X_s\right) ds \\
b) \quad & X_t = x + \int_0^t \exp(-X_s) dB_s + \frac{1}{2} \int_0^t \exp(-2X_s) ds \\
c) \quad & X_t = x + \frac{1}{3} \int_0^t (X_s)^{1/3} ds + \int_0^t (X_s)^{2/3} dB_s
\end{aligned}$$

Hint: assume that $X_t = \varphi(B_t)$ and use Ito formula to obtain an equation for φ.
In c) you can assume first that $X_t = \varphi(B_t + a(t))$ and after using Ito formula, choose the function $a(t)$ to simplify the differential equation for φ.

(b) Rewrite the SDE in Stratonovich form.

Remark in general is not always possible to find an explicit solutions of a SDE.

7. Let $B^{(1)}$ and $B^{(2)}$ two independent Brownian motions under the measure P and let

$$\begin{aligned}
X_t &= x^{(0)} t + x^{(1)} B_t^{(1)} + x^{(2)} B_t^{(2)} \\
Y_t &= y^{(0)} t + y^{(1)} B_t^{(1)} + y^{(2)} B_t^{(2)}
\end{aligned}$$

where $x^{(i)}, y^{(i)}$ are deterministic constants, $i = 0, 1, 2$.

Using Girsanov theorem, construct a probability measure Q equivalent to P on finite intervals $[0, t]$ such that both X_t and Y_t are Q-martingales.

Under which conditions on the coefficients $x^{(i)}, y^{(i)}$ such Q is unique ?

8. We consider a family of linear SDE in Ito sense

$$X_t = x + \int_0^t X_s \theta ds + \int_0^t X_s \sigma dB_s^\theta$$

where (B_t^θ) is Brownian motion under the measure P^θ. We think as $\sigma \neq 0$ fixed, while $\theta \in \mathbb{R}$ is a parameter. Note that

$$B_t^\theta = B_t^0 - \frac{\theta}{\sigma} t$$

where B_t^0 is a Brownian motion under P^0 which corresponds to the value $\theta = 0$.

a) Compute and the likelihood ratio process

$$Z_t(\theta) = \frac{dP_t^\theta}{dP_t^0}$$

and find a representation as stochastic integral with respect to the integrator (X_t).

b) Show that $Z_t(\theta)$ is a martingale under P^0.

c) Compute the logarithmic derivative

$$S_t(\theta) := \frac{d}{d\theta} \log Z_t(\theta)$$

and show that $S_t(\theta)$ is a martingale under P^θ.

d) Assuming now that the parameter θ is unknown, compute the maximum likelihood estimator $\hat{\theta}_T$ for a given a realization $(X_t(\omega) : t \in [0, T])$. In other words, find the argument $\hat{\theta}$ which maximizes $\log(Z_t(\theta, \omega))$ for the observed realization.