UH Stochastic analysis I, Spring 2017, Exercise-4 (14-21.3 2017)

We will discuss these exercises on tuesday 14.3 in exercise class from 10-12, room D123.

We recall the integration by parts formula for cadlag functions with finite variation:

$$X(t)Y(t) - X(0)Y(0) = \int_0^t X(s)Y(ds) + \int_0^t Y(s-)X(ds) = \tag{0.1}$$
$$\int_0^t X(s-)Y(ds) + \int_0^t Y(s)X(ds)$$
$$= \int_0^t X(s-)Y(ds) + \int_0^t Y(s-)X(ds) + [X,Y]_t$$

where $[X,Y]_t = \sum_{s\le t} \Delta X(s)\Delta Y(s)$ is the cross variation.

For cadlag functions $X(t)$ with finite variation on compacts and differentiable functions $f(x)$, we have the change of variable formula

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x}(X(s))X(ds) + \sum_{s\le t}\Big(f(X(s)) - f(X(s-)) - \frac{\partial f}{\partial x}(X(s-))\Delta X(s)\Big) . \tag{0.2}$$

Recall also that if $Y(t)$ is a $\mathbb{F}$-adapted cadlag process with integrable variation on compact intervals

$$E(\mathrm{Var}_Y(t)) = E\Big(\int_0^t |Y(ds)|\Big) < \infty \forall t,$$

then its dual $\mathbb{F}$-predictable projection Y^p exists and $M(t) := Y(t) - Y^p(t)$ is a $\mathbb{F}$-martingale. If $X(t)$ is $\mathbb{F}$-predictable and

$$E\Big(\int_0^t |X(s)|\,|Y(ds)|\Big) < \infty \forall t,$$

then

$$(X\cdot Y)^p = \Big(\int_0^{\cdot} X(s)Y(ds)\Big)^p = \int_0^{\cdot} X(s)Y^p(ds)$$

and

$$(X\cdot M)_t = \int_0^t X(s)M(ds) = \int_0^t X(s)Y(ds) - \int_0^t X(s)Y^p(ds).$$

is a $\mathbb{F}$-martingale. Note also that if $X(s)$ is a cadlag $\mathbb{F}$-adapted process, it is $\mathbb{F}$-optional and its left limit $X(s-) = \lim_{r\uparrow s} X(r)$ is $\mathbb{F}$-predictable.

1. (Discrete time embedded into continuous time). Consider in discrete time a process $(X_n : n \in \mathbb{N})$, and a discrete filtration, $(\mathcal{F}_n : n \in \mathbb{N})$, where X_n in not necessarily $\{\mathcal{F}_n\}$-measurable. Assume that X is integrable or more in general locally integrable in the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

   We imbed the discrete time processes and filtrations, into continous time processes $X(t)$ and filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0)$ which are right-continuous, and piecewise constant between the jump times $n \in \mathbb{N}$:

   $$\mathcal{F}_t = \mathcal{F}_n \text{ and } X(t,\omega) = X_n(\omega) \quad \forall t \in [n, n+1),$$

   Show that in continuous time, the $\mathbb{F}$-optional and $\mathbb{F}$-predictable projections of the imbedded process X are respectively

   $$^{o}X(t) = E\big(X_n \big| \mathcal{F}_n\big) \quad t \in [n, n+1) \text{ and } {}^{p}X(t) = {}^{o}X(t) \text{ if } t \notin \mathbb{N}, \text{ and } {}^{p}X(n) = E\big(X_n \big| \mathcal{F}_{n-1}\big)$$

   Show that the dual $\mathbb{F}$-optional projection of X is

   $$X^o_t = \sum_{0\leq n\leq t} E\big(X_n - X_{n-1} \big| \mathcal{F}_n\big) = \sum_{0\leq s\leq t} E\big(\Delta X(s) | \mathcal{F}_s\big)$$

   and the dual $\mathbb{F}$- predictable projection is

   $$X^p_t = \sum_{0\leq n\leq t} E\big(X_n - X_{n-1} \big| \mathcal{F}_{n-1}\big) = \sum_{0\leq s\leq t} E\big(\Delta X(s) | \mathcal{F}_{s-}\big) = \sum_{0\leq s\leq t} \lim_{r\uparrow s} E\big(\Delta X(s) | \mathcal{F}_r\big)$$

2. Consider a Bernoulli counting process in the discrete time filtration $\{\mathcal{F}_k : k \in N\}$

   $$N(k) = \sum_{i=1}^{k} X_k(\omega)$$

   where

   $$P\big(X_i = 1 \big| \mathcal{F}_{i-1}\big)(\omega) = 1 - P\big(X_i = 1 \big| \mathcal{F}_{i-1}\big)(\omega) = p \in (0,1)$$

   - Imbed now the Bernoulli process N and the filtration in continuous time, and compute the projections ${}^{o}N, {}^{p}N, N^o, N^p$.

- For $a > 0$ show that

$$a^{N_n} = 1 + \sum_{j=1}^{n} (a-1) a^{N_{j-1}} \Delta N_j.$$

and use this identity to compute $\mathbb{E} a^{N_n}$.

3. Recall that the process X is stochastically continuous, if for each t and $\epsilon > 0$:

$$\lim_{h \to 0} P\{|X_{t+h} - X_t| > \epsilon\} = 0.$$

Show that the Bernoulli process (imbedded to continuous time) is not stochastically continuous.

4. Show that any increasing process X, $X_t \in L(P)$, with continuous expectation is stochastically continuous. Continuous expectation means that the map $t \mapsto EX_t$ is continuous at each point t. Check, using this fact, that Poisson process is stochastically continuous.

5. A piecewise constant cadlag process $N(t)$ with $N(0) = 0$ and $\Delta N(t) \in \{0, 1\}$ $\forall t$ is called a *counting process*.

   (a) Use the integration parts formula (0.1) with $X(t) = Y(t) = N(t)$ to compute $(N_- \cdot N)_t$ and $(N \cdot N)_t$.

   (b) For the next questions we assume that N is $\mathbb{F}$-adapted and $P(N_t < \infty) = 1$ $\forall t \geq 0$. Show that an $\mathbb{F}$-adapted counting process is locally bounded. Hint: find a sequence of stopping times $\tau_n(\omega) \uparrow \infty$ such that $N_{\tau_n \wedge t}(\omega) \leq C_n$ $\forall \omega, t$, with constants $C_n < \infty$.

   (c) Assume that N is $\mathbb{F}$-adapted and $P(N_t < \infty) = 1$, and show that the dual predictable projection N^p exists, and $\Delta N^p(t) = N^p(t) - N^p(t-) \in [0, 1]$.

   (d) Show also that

$$P\{N_t \geq \epsilon\} \leq EA_t, \forall\, \epsilon > 0.$$

   (e) Prove that $N(t)$ is stochastically continuous if and only if $t \mapsto N^p(t)$ is continuous.

6. Assume that $(Y_k)k \geq 1$, with $Y_0 = 0$, is a sequence of independent Bernoulli random variables with parameter p_k: $P\{Y_k = 1\} = p_k$. Define a counting process N on $[0, 1)$ by

$$N_t \doteq \sum_{k=0}^{\lfloor \frac{1}{1-t} - 1 \rfloor} Y_k$$

and put $N_1 = \lim_{t\to 1} N_t$. Show that the process N is <u>non-exploding</u> at the time $t = 1$ if and only if $\sum_k p_k < \infty$. Non-exploding: $P(N_1 < \infty) = 1$. Show also that if N is non-exploding at $t = 1$, then $EN_1 < \infty$.

7. Assume that F is continuous and f is right-continuous with bounded variation. Show that $t \mapsto F(f(t))$ is right-continuous. If $F \in C^1$, i.e. F is differentiable with a continuous derivative, then $F \circ f$ has bounded variation on compacts

8. Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, with cadlag trajectories and a filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ such that $M(t) = (N(t) - \lambda t)$ is a $\mathbb{F}$-martingale.
   (a) What are the $\mathbb{F}$-optional and $\mathbb{F}$-predictable projections ${}^oN$ and ${}^oN$?
   (b) Show that $N(t)$ has dual $\mathbb{F}$-optional and dual $\mathbb{F}$-predictable projections, find N^o and N^p.
   (c) Let $N(t)$ be the Poisson process in a filtration $\mathbb{F}$ as in Exercise 1. Use inductively the integration by part formula (0.1) with $X(t) = N(t)^{k-1}$, $Y(t) = N(t)$ to compute the moments $E(N(t)^k)$.

9. We compute the Laplace transform of the λ-Poisson process using martingales.

   For $\theta > 0$ let $f(x) = \exp(-\theta x)$, and use the change of variable formula for cadlag functions $X(t)$ with finite variation on compacts and differentiable $f(x)$

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x}(X(s))X(ds) + \sum_{s\leq t}\big(f(X(s)) - f(X(s-)) - \frac{\partial f}{\partial x}(X(s-))\Delta X(s)$$

   and a martingale argument to compute the Laplace transform $\theta \mapsto E\big(\exp(-\theta N(t))\big)$, $\theta > 0$ of the λ-Poisson process $N(t)$.

10. **A theorem by Thomas Kurtz** In this exercise we use a martingale argument together with the change of variable formula (0.2) to compute Laplace transforms.

    Let $\tau_1, \ldots \tau_m$ $\mathbb{F}$-stopping times, and let $N_j(t) = \mathbf{1}(\tau_j \leq t)$. Let $\Lambda_j = N_j^p$, the compensator or dual $\mathbb{F}$-predictable projection of N_j (which exists, why ?)

    We assume that

$$P(\tau_i = +\infty) = P(\tau_i = \tau_j) = 0 \quad \forall i \neq j$$

and that the compensators $\Lambda_j(t)$ are continuous processes.

We show that the stopped compensators $\Lambda_1(\tau_1), \ldots, \Lambda_m(\tau_1)$ are i.i.d. 1-exponential random variables, i.e.

$$P\big(\Lambda_1(\tau_1) > x_1, \ldots, \Lambda_1(\tau_1) > x_n\ \big) = \exp\Big(-\sum_{j=1}^{m} x_j \Big), \quad x_j > 0$$

In order to show it we compute the joint Laplace transform of $\Lambda_{\tau_1}, \ldots \Lambda_{\tau_m}$ and show that for $\forall \theta_j > 0$

$$E\bigg(\exp\Big(-\sum_{j=1}^{m} \theta_1 \Lambda_j(\tau_i) \Big) \bigg) = \prod_j \frac{1}{(1+\theta_j)} \tag{0.3}$$

which is the product of the Laplace transform of i.i.d. 1-exponential random variables.

(a) Use the change of variable formula to write an integral representation of

$$\zeta_j(\theta_j, t) = (1+\theta_j)^{N_j(t)} \exp\big(-\theta_j \Lambda_j(t)\big)$$

and show that if $\theta_j > 0$, $\zeta_j(t)$ is an uniformly integrable F-martingale.

(b) Show that

$$[\zeta_i(\theta_i), \zeta_j(\theta)]_t = \sum_{s\le t} \Delta\zeta_i(\theta_i, s)\Delta\zeta_j(\theta_j, s) = 0 \quad \forall i \neq j$$

Hint:

$$[N_i, N_j]_t = \sum_{s \le t} \Delta N_i(t) \Delta N_j(t) = 0 \quad \forall i \neq j$$

(c) Use the integration by part formula for product of finite variation processes, to find an integral representation for the product

$$Z(\theta, t) = \prod_{j=1}^{m} \zeta_j(\theta_j, t)$$

and show that $\forall \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}_+^m$, $Z(\theta, t)$ is also an uniformly integrable martingale.

(d) Compute $E\big(Z(\theta, \infty)\big)$ to prove (0.3)