UH Stochastic analysis I, Spring 2017, Exercise-2 (8.2 2017)

In all exercises $(B_t : t \geq 0)$ is a Brownian motion on the probability $(\Omega, \mathcal{F}_t, P)$, in the filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$, which means that B is $\mathbb{F}$-adapted and for $0 \leq s \leq t$, $(B_t - B_s) \perp\!\!\!\perp \mathcal{F}_s$ w.r.t. P.

1. (a) Show that the Riemann integral

$$A_t = \int_0^t B_u(\omega) du$$

is a Gaussian random variable and compute its mean and variance. Hint: compute the Gaussian law of a Riemann sum approximation and take limit in distribution.

(b) Show by a direct computation that

$$M_t = B_t^3 - 3 \int_0^t B_u du$$

is a martingale in the Brownian filtration. Hint: for $0 \leq s \leq t$,

$$M_t = \{B_s + (B_t - B_s)\}^3 - 3 \int_0^s B_u du - 3(t-s)B_s - 3 \int_s^t (B_u - B_s) du$$

2. Let $(\tau_n(\omega) : n \in \mathbb{N})$ a sequence of stopping times in the $\mathbb{F}$ filtration. Show that

$$\tau_*(\omega) = \inf_n \{ \tau_n(\omega)\}, \quad \tau^* = \sup_n \{ \tau_n(\omega)\}$$

are $\mathbb{F}$-stopping times.

3. Consider two filtrations $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ and $\mathbb{G} = \{\mathcal{G}_t : t \geq 0\}$ on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that $\mathbb{F} \subseteq \mathbb{G}$ meaning that $\mathcal{F}_t \subseteq \mathcal{G}_t$ $\forall t$.

Show that if M_t is a $(P, \mathbb{G})$-martingale and it is also $\mathbb{F}$-adapted, then it is a martingale also with respect to the smaller filtration $\mathbb{F}$.

4. Let B_t, W_t $\mathbb{P}$-independent Brownian motion (on the same probability space). We define their cross-variation (or quadratic covariation as)

$$[B, W]_t := \lim_{\Delta(\Pi^n)} \sum_k \big(B_{t_k^n \wedge t} - B_{t_{k-1}^n \wedge t}\big)\big(W_{t_k^n \wedge t} - W_{t_{k-1}^n \wedge t}\big) \qquad (0.1)$$

where the limit is taken in $L^2(P)$ for a sequence of partitions $\Pi^n = \{0 = t_0^n < t_1^n < \cdots < t_k^n < \dots\}$ where $\lim_{k\to\infty} t_k^n \to \infty$ and

$$\Delta(\Pi^n) = \sup_k |t_k^n - t_{k-1}^n| \to 0 \text{ as } n \to \infty$$

(a) Show that the $L^2(P)$ limit is $[B, W]_t = 0$ independently of the choice the sequence of partitions $\{\Pi^n\}$.

(b) Show that if the sequence of partitions is refining, i.e. $\Pi^n \subseteq \Pi^{n+1}$ $\forall n$, then we have also P-almost sure convergence in (0.1)

(c) Show by direct calculation that the product $M_t = B_t W_t$ is a martingale in the Brownian filtration

Hint: as in the proof of the result for the quadratic vatiation of Brownian motion $[B]_t = t$,

5. Let $\tau(\omega) \in [0, +\infty]$ be a random time, $F(t) = P(\tau \le t)$ for $t \in [0, \infty)$.

Consider the single jump *counting process* $N_t(\omega) := \mathbf{1}(\tau(\omega) \le t)$ which generates the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t^N = \sigma(N_s : s \le t)$.

(a) Show that τ is a stopping time in the filtration $\mathbb{F}$.

(b) Show that first that for every Borel function $f(x)$, the random variable

$$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s)$$

is $\mathcal{F}_s$-measurable.

(c) Define the *cumulative hazard function*

$$\Lambda(t) = \int_0^t \frac{1}{1 - F(s-)} F(ds)$$

where $F(s-) = P(\tau < s)$ denotes the limit from the left.

Show that

$$M_t = N_t - \Lambda_{t \wedge \tau}$$

is a an $\mathbb{F}$-martingale.

Hint: use the definition, and show that for $s \le t$ and every $A \in \mathcal{F}_s$

$$E_P\Big((N_t - N_s)\mathbf{1}_A \Big) = E_P\Big((\Lambda_{t\wedge\tau} - \Lambda_{s\wedge\tau})\mathbf{1}_A \Big)$$

It turns out that it is enough to do the computation for $A = \{\omega : \tau(\omega) > s\}$ (why ?). Fubini's theorem may be also useful.

(d) Assume that $t \mapsto F(t)$ and therefore also $t \mapsto \Lambda(t)$ are continuous, which means $P(\tau = t) = 0 \; \forall t \in \mathbb{R}^+$.

Show that Λ_τ has 1-exponential distribution:

$$P\big(\Lambda_\tau > x\big) = \exp(-x), \quad x \geq 0$$

Hint: one line of proof compute the Laplace transform

$$\mathcal{L}(\theta) := E_P\Big(\exp(-\theta\Lambda_\tau)\Big) \quad \theta > 0$$

and compare it with the Laplace transform of the 1-exponential distribution.

(e) Show that the martingale M_t is uniformly integrable, what is M_∞?.

6. Let $\theta \in \mathbb{R}$, and $i = \sqrt{-1}$ the imaginary unit

Show that

$$E_P\big(\exp(i\theta B_t)\big) = \exp\big(-\frac{1}{2}\theta^2 t\big)$$

Hint: Use complex integration over the rectangular contour delimited by in the complex plane by the points $R, (R + i\theta), (-R + i\theta), -R$ with $R \in \mathbb{R}$ and let $R \to \infty$.

7. For $\theta \in \mathbb{R}$, consider now

$$M_t = \exp\big(i\theta B_t + \frac{1}{2}\theta^2 t\big) = \Big\{\exp\big(\frac{1}{2}\theta^2 t\big)\cos(\theta B_t) + \sqrt{-1}\exp\big(\frac{1}{2}\theta^2 t\big)\sin(\theta B_t)\Big\} \in \mathbb{C}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Recall that $E(\exp(i\theta G)) = \exp(-\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0, \sigma^2)$.

- Show that M_t is complex valued $\mathbb{F}$-martingale, which means that real and imaginary parts are $\mathbb{F}$-martingales.
- Show that $\lim_{t\to\infty} |M_t(\omega)| = \infty$